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# The Theory of the Concave Grating

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THE theory of the concave grating was developed, to a large degree, by Rowland himself;[1] later contributions were made by Glazebrook,[2] Mascart,[3] Baily,[4] and, to an important degree, by Runge.[5] The use of the concave grating in grazing incidence has been investigated by Mack, Stehn, and Edlén,[6] and by Bowen.[7] Numerous other contributions to special aspects of grating performance, such as astigmatism and aberration, will be cited when these subjects are discussed.

In the present article, a general theory of the image formation of the concave grating, based on Fermat's principle, will be developed. Such a discussion should be useful in the consideration of questions that arise on the adjustment of concave gratings. For the images formed by the concave grating are subject to the inherent imperfections of all image formation, and the magnitude of these imperfections must be calculated so that they may be distinguished from the effects of imperfections in ruling and in the figure of the grating itself which are always present to a greater or less extent. Furthermore since gratings are individually ruled, and are usually mounted and adjusted in the research laboratory, there is greater need for information as to their adjustment than there is in the case of prism spectrographs, which are most often tested and furnished by the makers as complete instruments. The usefulness of a knowledge of the general theory of the grating, therefore, seems likely to persist for some time.

## I. THE CONDITIONS FOR A FOCUS

In Fig. 1, light coming from a point $A$ falls on the point $P$, on the surface of a concave grating, and is diffracted to the point $B$. The origin, $O$, of the cartesian coordinate system is taken in the center of the grating surface, with the $x$-axis horizontal, and the $z$-axis vertical and parallel to

* This paper was nearly completed at the time of Dr. Beutler's death on December 17, 1942. It has been prepared for publication by Commander R. A. Sawyer, U.S.N.R., Naval Proving Ground, Dahlgren, Virginia. Although Dr. Beutler had intended to enlarge some parts of this paper, it is believed to be sufficiently complete to be of interest.

[1] H. A. Rowland, Phil. Mag. **13**, 469 (1882); **16**, 197 and 210 (1883).

[2] R. T. Glazebrook, Phil. Mag. **15**, 414 (1883).

[3] E. Mascart, J. de Physique **2**, 5 (1883).

[4] W. Baily, Phil. Mag. **22**, 47 (1886).

[5] C. Runge, in Kayser's *Handbuch*, Vol. I, p. 450–470; C. Runge and K. W. Meissner, *Handbuch der Astrophysik*, (Verlagsbuchhandlung Julius Springer, Berlin, 1933). Vol. I, Pp. 235–257.

[6] J. E. Mack, J. R. Stehn, and B. Edlén, J. Opt. Soc. Am. **22**, 245 (1932).

[7] I. S. Bowen, J. Opt. Soc. Am **23**. 313 (1933).

the grating rulings. Points in the light source $A$ (i.e., of the slit of the spectrograph) are designated by the coordinates $x$, $y$, $z$; points of the image or spectral line, by $x'$, $y'$, $z'$, and points on the grating surfaces, by $\xi$, $w$ (width) and $l$ (length of rulings measured from the center). The coordinate $\xi$, parallel to the $x$-axis, gives the distance of any point on the surface of the grating blank from the $l$–$w$ plane.

The light path is, then, the two straight lines, $AP$ and $BP$, whose lengths are expressed in rectangular coordinates as

$$\langle AP\rangle^2=(x-\xi)^2+(y-w)^2+(z-l)^2; \qquad (1)$$
$$\langle BP\rangle^2=(x'-\xi)^2+(y'-w)^2+(-z'-l)^2.$$

The signs of $z$, and $z'$ have been chosen oppositely to account for the fact that, because of reflection at any grating point, $P$, the points $A$ and $B$ will lie on different sides of the $XY$-plane. $A$ and $B$ do not, however, necessarily lie on opposite sides of the $XZ$-plane, since diffraction, as well as reflection, occurs in the $y$-direction. Thus the signs of $y$ and $y'$ are independent of one another. The $x$-coordinate is always positive for reflection gratings.

For practical purposes it is convenient to express Eqs. (1) in terms of the distances between the points $A$ and $B$ and the grating center, $O$, and of the angles of incidence and diffraction, both measured in the $x$–$y$ plane from $O$ to the projections of $A$ and $B$. Hence, cylindrical coordinates with their origin in the center of the grating are introduced, $(r, \alpha)$ for $A$, and $(r', \beta)$ for $B$ as follows:

$$x=r\cos\alpha, \quad x'=r'\cos\beta; \qquad (2)$$
$$y=r\sin\alpha, \quad y'=r'\sin\beta.$$

The signs of $\alpha$ and $\beta$ are dependent on those of $y$ and $y'$ respectively, and are opposite if $A$ and $B$ lie on different sides of the grating normal.

All points, $P$, of the grating surface lie on a sphere of radius $R$ and so must satisfy the equation,

$$(R-\xi)^2+w^2+l^2-R^2=0, \qquad (3)$$

whence

$$\xi^2-2R\xi+w^2+l^2=0. \qquad (3a)$$

By application of the formula for the roots of a quadratic equation

$$\xi=R\pm[R^2-(w^2+l^2)]^{\frac{1}{2}}.$$

Only the negative value of the radical is significant, since the positive value leads to the points on the opposite end of the diameter. Expanding the solution, with the negative radical, in a power series

$$\xi=\frac{w^2+l^2}{2R}+\frac{(w^2+l^2)^2}{8R^3}+\frac{(w^2+l^2)^3}{16R^5}+\frac{5(w^2+l^2)^4}{128R^7}+\cdots. \qquad (4)$$

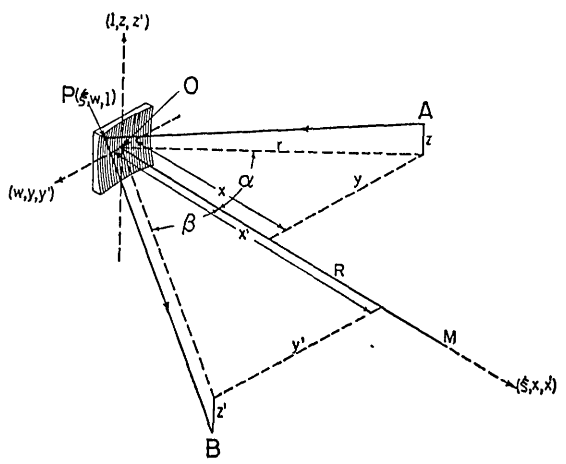


FIG. 1. Image formation by the concave grating.

The "rulings" on the reflecting grating are grooves that have been cut or "ruled" into the blank concave mirror. Rowland[1] showed that they should be so spaced on the spherical surface as to be equally spaced on the chord of the circular arc in the $x$, $y$ plane; i.e., they are the traces of equidistant planes perpendicular to the $w$-co-ordinate. The reflectivity of the grating surface varies periodically with $w$ because of the grooves. For most purposes it is sufficient to represent the diffracting areas by simple lines, equally spaced and located at the centers of light intensity of the actual areas. Details of phase variation of the light reflected from various points across the areas are thereby ignored. Actually these phase variations do exist and depend on both the shape and the spacing of the rulings as well as on the angles of incidence and diffraction. The phase variations are of importance in considerations of the intensity distribution in the spectrum and of the relative intensity of different orders.

To introduce into the equations a set of equally spaced lines as rulings, it is sufficient to define the co-ordinate $w$ as having only discrete equidistant values fixed by the grating constant, $d$ (the distance between the rulings). Thus $w/d$ is a pure number which is permitted to have only integral values, counting from the grating center to either side. As in the case of the plane grating, the light path for neighboring grooves must differ by one wave-length, or by an integral multiple of one wave-length, in order that the rays may reinforce one another. That is, the path difference for any two grooves separated by $w$ is $(w/d)m\lambda$, where $m$ is an integer called the order number. If light from any point, $P$, is to contribute to the image, light originating in point $A$ and focused in $B$, must satisfy the light path function, $F$,

$$F=AP+BP+(w/d)m\lambda. \tag{5}$$

This function is a characteristic function in the sense of Hamilton;[8] it represents the permissible lengths of the various lightpaths from $A$ to $B$, as the point $P$ wanders over the ruled surface of the grating. The various rays must arrive at $B$ with the same phase in order to produce reinforcement of the amplitudes. Therefore Eq. (5) must be fulfilled within the limits of a quarter wavelength by Rayleigh's criterion.

According to Fermat's principle of least time, point $B$ is located so that this function will be an extreme for any point, $P$; and to focus $A$ at $B$, all these extremes for the various points, $P$, must be equal. The conditions for focusing the light from different parts of the grating may be separated. Thus to get a focus for light from points along $w$ in the diffracting surface of the grating, the condition for equal extremes of the light paths is that the partial derivative of $F$ with respect to $w$ be zero, or

$$\partial F/\partial w=0. \tag{6a}$$

Likewise a focus for the light from any vertical section of the grating requires

$$\partial F/\partial l=0. \tag{6b}$$

These partial derivatives have the geometrical significance of angles. If their values are zero, then the direction of the light beam coming from $A$ and reflected at points $P$ along $w$ or $l$, respectively, is strictly towards $B$. Any deviation of the partial derivatives from zero indicates that some light falls outside of $B$, and that the image formation is not perfect. The displacement of such light sidewise from the focus can be calculated readily: its distance will be $r'\partial F/\partial w$ (or $r'\partial F/\partial l$, respectively) from the point $B$ in a plane that is perpendicular to the beam at $B$. If the normal to the focal plane is inclined to the beam by the angle $\gamma$, then in this inclined plane the light will fall at the distance $\Delta_p$ from $B$, given by:

$$\Delta_p=r'(\partial F/\partial l)(1/\cos\gamma). \tag{6c}$$

The fact that the image at the point $B$ is in reality a diffraction pattern allows a range of $\Delta_p$ within the width of the central maximum of the diffraction figure without much loss in resolving power.

In addition to the loss of light at the focal point, there is another loss in intensity because of light that reaches $B$ with the wrong phase. If a part of the grating along the $w$-coordinate is not at the proper distance from the points $A$ and $B$, then the partial derivative $\partial F/\partial w$ will not be zero; the path difference in the light beam accruing over the length $dw$ will amount to $\Delta F=\int(\partial F/\partial w)\cdot dw$. The condition for reinforcement of the light beams is that over the entire grating the value of this integral does not exceed the value $\pm\lambda/4$; the necessary condition is:

$$\Delta F_w=\int_{-W/2}^{+W/2}\frac{\partial F}{\partial w}\cdot dw\leqslant\frac{\lambda}{4}. \tag{6d}$$

($W$ is the full width of the grating).

For the $l$-coordinate and its respective focus, a similar equation holds:

$$\Delta F_l=\int_{-L/2}^{+L/2}\frac{\partial F}{\partial l}\cdot dl\leqslant\frac{\lambda}{4}. \tag{6e}$$

($L$ is the full length of the rulings.) Equation (6a) is necessary for the general focusing conditions of the grating; (6c) for the quantitative treatment of coma and curvature; and Eq. (6d) will be used to derive the spherical aberration; accordingly, Eq. (5) may be written more exactly:

$$F=AP+BP+(w/d)m\lambda\pm\lambda/4. \tag{5b}$$

[8] W. R. Hamilton, *Mathematical Papers*. (The University Press, Cambridge, England, 1931). Vol. I, p. 17

## II. THE GENERAL EQUATIONS

The formulas (1) to (6) express all the facts about the optics of the grating, except those for groove form, that lead to the intensity relations. For an analysis of the geometrical optics of the concave grating, these expressions must be related and evaluated. The series expansions, although somewhat cumbersome, will be used because of the significance that can be attached to the various terms. The expressions for $AP$ and $BP$ are identical in form, except for the primed coordinates in the latter, and therefore only the derivations for $AP$ are presented.

$$\begin{aligned}\langle AP\rangle^2 &= (x-\xi)^2+(y-w)^2+(z-l)^2;\\ &= x^2+y^2+z^2+\xi^2+w^2+l^2-2x\xi-2yw-2zl.\end{aligned} \tag{1}$$

Introducing expressions (2) and (3a),

$$\langle AP\rangle^2 = r^2+z^2+2R\xi-2x\xi-2yw-2zl.$$

Using expressions (2) and (4) for $x$, $y$ and $\xi$,

$$\langle AP\rangle^2 = r^2+z^2-2r\cdot w\cdot \sin\alpha-2z\cdot l+\left(1-\frac{r}{R}\cdot\cos\alpha\right)\left(w^2+l^2+\frac{(w^2+l^2)^2}{4R^2}+\frac{(w^2+l^2)^3}{8R^4}+\cdots\right)$$

$$= r^2-2rw\cdot\sin\alpha+w^2\sin^2\alpha+w^2\cos^2\alpha-\frac{r}{R}w^2\cos\alpha+l^2\left(1-\frac{r}{R}\cos\alpha\right)$$

$$-2zl+z^2+\left(1-\frac{r}{R}\cdot\cos\alpha\right)\left(\frac{(w^2+l^2)^2}{4R^2}+\frac{(w^2+l^2)^3}{8R^4}+\cdots\right).$$

Whence

$$\langle AP\rangle^2 = (r-w\sin\alpha)^2+w^2\left(\cos^2\alpha-\frac{r}{R}\cos\alpha\right)+l^2\left(1-\frac{r}{R}\cos\alpha\right)-2lz+z^2$$

$$+\frac{(w^2+l^2)^2}{4R^2}\left(1-\frac{r}{R}\cos\alpha\right)+\frac{(w^2+l^2)^3}{8R^4}\left(1-\frac{r}{R}\cos\alpha\right)+\cdots. \tag{7}$$

The square root can be approximated by series development, whence

$$AP = r-w\cdot\sin\alpha+\frac{\frac{1}{2}w^2(\cos^2\alpha-(r/R)\cos\alpha)}{r-w\sin\alpha}+\tfrac{1}{2}l^2\frac{(1-(r/R)\cdot\cos\alpha)}{r-w\sin\alpha}$$

$$+\tfrac{1}{2}\frac{-2lz+z^2}{r-w\sin\alpha}+\frac{(w^2+l^2)^2}{8R^2}\frac{(1-(r/R)\cos\alpha)}{(r-w\sin\alpha)}-\frac{w^4(\cos^2\alpha-(r/R)\cos\alpha)^2}{8(r-w\sin\alpha)^3}+\cdots \tag{8}$$

and by further series development the final expression is:

$$AP = r-w\sin\alpha+\tfrac{1}{2}w^2\left(\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}\right)+\tfrac{1}{2}l^2\left(\frac{1}{r}-\frac{\cos\alpha}{R}\right)-\frac{lz}{r}+\frac{z^2}{2r}+\cdots$$

$$+\frac{(w^2+l^2)^2}{8R^2}\left(\frac{1}{r}-\frac{\cos\alpha}{R}\right)-\cdots+\tfrac{1}{2}w^3\frac{\sin\alpha}{r}\left(\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}\right)+\tfrac{1}{2}l^2\frac{w\sin\alpha}{r}\left(\frac{1}{r}-\frac{\cos\alpha}{R}\right)$$

$$+\frac{w\sin\alpha}{2r^2}\cdot(-2lz+z^2)+\frac{(w^2+l^2)^2}{8R^2}\frac{w\sin\alpha}{r}\left(\frac{1}{r}-\frac{\cos\alpha}{R}\right)-\cdots+\tfrac{1}{2}w^4\frac{\sin^2\alpha}{r^2}\left(\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}\right)$$

$$+\tfrac{1}{2}l^2\frac{w^2\sin^2\alpha}{r^2}\left(\frac{1}{r}-\frac{\cos\alpha}{R}\right)+\frac{w^2\sin^2\alpha}{2r^3}(-2lz+z^2)-\frac{w^4}{8r^2}\left(\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}\right)^2+\cdots. \tag{9}$$

In order to simplify the discussion of this very cumbersome expression, write:

$$AP = F_1 + F_2 + F_3 + F_4 + F_5 + \cdots \tag{10}$$

and similarly for the diffracted beam:

$$BP = F_1' + F_2' + F_3' + F_4' + F_5' + \cdots. \tag{11}$$

For further use in the following sections, the expressions for $AP$ and $BP$ are tabulated separately:

$$F_1 = r - w \sin \alpha, \tag{10a}$$

$$F_2 = \tfrac{1}{2}w^2\left(\frac{\cos^2 \alpha}{r} - \frac{\cos \alpha}{R}\right) + \tfrac{1}{2}w^3 \frac{\sin \alpha}{r}\left(\frac{\cos^2 \alpha}{r} - \frac{\cos \alpha}{R}\right) + \tfrac{1}{2}w^4 \frac{\sin^2 \alpha}{r^2}\left(\frac{\cos^2 \alpha}{r} - \frac{\cos \alpha}{R}\right) + \cdots, \tag{10b}$$

$$F_3 = \tfrac{1}{2}l^2\left(\frac{1}{r} - \frac{\cos \alpha}{R}\right) - \frac{lz}{r} + \frac{z^2}{2r}, \tag{10c}$$

$$F_4 = \tfrac{1}{2}l^2 w \frac{\sin \alpha}{r}\left(\frac{1}{r} - \frac{\cos \alpha}{R}\right) + \frac{w \sin \alpha}{2r^2}(2lz + z^2), \tag{10d}$$

$$F_5 = \frac{(w^2+l^2)^2}{8R^2}\left(\frac{1}{r} - \frac{\cos \alpha}{R}\right), \tag{10e}$$

$$F_6 = \frac{-w^4}{8r^2}\left(\frac{\cos^2 \alpha}{r} - \frac{\cos \alpha}{R}\right)^2 - \frac{3w^5 \cdot \sin \alpha}{8r^3}\left(\frac{\cos^2 \alpha}{r} - \frac{\cos \alpha}{R}\right)^2, \tag{10f}$$

$$F_7 = \frac{(w^2+l^2)^2}{8R^2} \cdot \frac{w \sin \alpha}{r}\left(\frac{1}{r} - \frac{\cos \alpha}{R}\right). \tag{10g}$$

$$F_1' = r' - w \sin \beta, \tag{11a}$$

$$F_2' = \tfrac{1}{2}w^2\left(\frac{\cos^2 \beta}{r'} - \frac{\cos \beta}{R}\right) + \tfrac{1}{2}w^3 \frac{\sin \beta}{r'}\left(\frac{\cos^2 \beta}{r'} - \frac{\cos \beta}{R}\right) + \tfrac{1}{2}w^4 \cdot \frac{\sin^2 \beta}{r'^2}\left(\frac{\cos^2 \beta}{r'} - \frac{\cos \beta}{R}\right) + \cdots, \tag{11b}$$

$$F_3' = \tfrac{1}{2}l^2\left(\frac{1}{r'} - \frac{\cos \beta}{R}\right) - \frac{lz'}{r'} + \frac{z'^2}{2r'}, \tag{11c}$$

$$F_4' = \tfrac{1}{2}l^2 w \frac{\sin \beta}{r'}\left(\frac{1}{r'} - \frac{\cos \beta}{R}\right) + \frac{w \cdot \sin \beta}{2r'^2}(-2lz' + z'^2), \tag{11d}$$

$$F_5' = \frac{(w^2+l^2)^2}{8R^2}\left(\frac{1}{r'} - \frac{\cos \beta}{R}\right), \tag{11e}$$

$$F_6' = -\frac{w^4}{8r'^2}\left(\frac{\cos^2 \beta}{r'} - \frac{\cos \beta}{R}\right)^2 - \frac{3w^5 \sin \alpha}{8r'^3}\left(\frac{\cos^2 \beta}{r'} - \frac{\cos \beta}{R}\right)^2, \tag{11f}$$

$$F_7' = \frac{(w^2+l^2)^2}{8R^2} \cdot \frac{w \cdot \sin \beta}{r'}\left(\frac{1}{r'} - \frac{\cos \beta}{R}\right). \tag{11g}$$

These somewhat cumbersome appearing expressions are actually convenient and of physical significance for the treatment of the grating theory. The order of the terms is that of decreasing magnitude for all cases except those of very large angles of incidence, $\alpha$, and of diffraction, $\beta$. It is accordingly easy to estimate the errors that are introduced by the neglect of any of the higher terms of the expansion. Furthermore, most of the expressions have individual physical significance, either as regards the image formation or its imperfections. The terms $(F_1+F_1')$, and $(F_2+F_2')$ give the condition for image formation for both the plane and the concave grating. $(F_2+F_2')$ can and must be made zero for a good focus. $(F_3+F_3')$ gives the astigmatism and $(F_4+F_4')$ the coma in the image and the curvature of the spectral lines. In general neither fault can be avoided completely. $(F_5+F_5')$ determines the spherical aberration, thereby limiting the useful size of the grating for any particular values of $\alpha$, $\beta$, $R$, and $d$. $(F_6+F_6')$ can be kept equal to zero, and $(F_7+F_7')$ represents a higher order aberration, which can usually be kept negligible in comparison to $(F_5+F_5')$.

The decreasing order of magnitude of the successive expressions is seen from the fact that they contain successively higher inverse powers of $R$, $r$ or $r'$,—all of which are usually quite large in comparison to the grating width, $w$, or length, $l$. The numerical values of the expressions are given below for a 20 foot concave grating with a ruled surface six by three inches. It is assumed that incidence is nearly normal ($\alpha$ and $\beta$ less than 50°) and therefore the trigonometric functions of $\alpha$ and $\beta$ can be neglected as of a lower order of magnitude than the other terms.

The results are:

$F_1+F_1' \sim 800$ cm,

$F_2+F_2' \sim 5$ cm (can be made zero),

$F_3+F_3' \sim 2$ cm (astigmatism),

$F_4+F_4' \sim .01$ cm (coma and curvature of lines),

$F_5+F_5' \sim 10^{-4}$ cm (aberration),

$F_6+F_6' \sim 10^{-5}$ cm (can be made zero),

$F_7+F_7' \sim 5\times 10^{-6}$ cm.

The expression "can be made zero" indicates that these terms vanish in certain mountings of the concave grating, i.e., for certain definite relative positions of the light source and image with respect to the grating. The most important mountings having these good optical properties are those which use the Rowland circle or which use the region in the neighborhood of the grating normal only.

An advantage of the development of the characteristic function, $F$, in a sequence of members of rapidly decreasing magnitude lies in the ease with which the partial derivative of $F$ may be obtained. If $\partial(F_1+F_2+F_3)/\partial w=0$ has to be calculated, and $F_1$, $F_2$, $F_3\cdots$ are of very different magnitudes, then it is convenient to make $\partial F_1/\partial w=0$, before investigating $\partial F_2/\partial w=0$. Both must be satisfied simultaneously before $\partial F_3/\partial w$ is to be taken into account. The expressions form a set of separable conditions the physical significance of which can be readily understood.

The characteristic grating equation $F$ (Eq. (5)) becomes, by the introduction of expressions (10) and (11),

$$F=F_1+F_2+F_3+\cdots F_1'+F_2'+F_3' +\cdots+w/d\cdot m\lambda. \quad (12)$$

This equation contains the expressions of (10a)$\cdots$(10g) and (11a)$\cdots$(11g) and is to be subjected to the operations of (6a), (6b), and (6c) in the following sections.

### III. THE GRATING EQUATION

Using only the first terms of the expansion of $F$ and the last term in (12),

$$F_1^0=F_1+F_1'+w/d\cdot m\lambda. \quad (13a)$$

Introducing the expressions (10a) and (11a)

$$\begin{aligned} F_1^0&=r-w\cdot\sin\alpha+r'-w\sin\beta+w/d\cdot m\lambda \quad (13)\\ &=r+r'-w(\sin\alpha+\sin\beta)+w/d\cdot m\lambda. \end{aligned}$$

Applying Fermat's principle (6a), $\partial F/\partial w=0$,

$$(1/d)m\lambda=\sin\alpha+\sin\beta. \quad (14)$$

This expression will be recognized as the well-known equation for the plane grating $m\lambda=d(\sin\alpha+\sin\beta)$ which is obtained here as the first approximation. It follows from the postulate that neighboring grooves give rise to spectral lines if the various light paths differ in length by $\lambda$ or an integral multiple of $\lambda$.

It will be seen that this expression holds exactly for the plane grating, because for the plane grating, $R$ is infinite, and so also are $r$ and $r'$ since there is no focusing effect. Thus all the higher members, $F_2, F_2' \cdots F_7, F_7'$ are exactly zero.

On the other hand, Eq. (14) holds for the concave grating too. It gives the condition for the focusing of wave-lengths at particular angles of incidence and diffraction, and the rule for the overlapping of different orders. The quantitative evaluation of Eq. (14) is shown graphically in Fig. 2, which gives the dependence of the diffracted wave-length on the angles of incidence and diffraction for a grating with 30,000 lines per inch (or more exactly 12,000 lines per cm) and in the first order.

In order to find the wave-lengths obtained in the second order (or in general in the $m$th order), it is necessary only to divide all the values in the chart by two (in general, by $m$). If a 15,000 lines-per-inch grating is used, the wave-lengths of Fig. 2 are those which will appear in the second order, while the first-order wave-lengths will have double the values of the chart. Accordingly, the shape of the curves of equal wave-length, which may be called isochromats, is the same for any grating spacing or order. They are simply the graphs of the angular function $(\sin\alpha+\sin\beta)$. The wave-length values to be attributed to these curves follow from the equation, $m\lambda\cdot 1/d=$constant, and can easily be calculated in any given case.

## IV. THE MOUNTINGS OF THE CONCAVE GRATING

Before further development of the theory is undertaken, the most common mountings of the concave grating will be discussed briefly, so that their optical properties can be referred to in the later discussion

It is evident from Fig. 2 that the same wave-length may often be obtained at various angles of incidence and diffraction. For example, 6000A in the first order, or 3000A in the second, can be observed at

| $\alpha=80°$ | 70° | 60° | 50° | 46° | 40° | 30° | 21.3° | 20° | 10° |
|---|---|---|---|---|---|---|---|---|---|
| $\beta=-15.5°$ | $-13°$ | $-8.5°$ | $-3°$ | 0 | 4.5° | 13° | 21.3° | 23° | 33°. |

The question arises as to which of these positions is best. There is no simple answer. As long as only a limited range of wave-lengths is wanted, the answer depends on the qualities desired in the photograph. An optimum resolution may be wanted at a particular wave-length, in which case the effect on the aberrations of the various possible angles of incidence and diffraction must be studied; or a maximum of intensity may be wanted in some region, and in that case the astigmatism and the shape and spacing of the grooves in the grating to be used are important for the choice of $\alpha$ and $\beta$. In general, however, the requirement is not to favor one property at the expense of all others, and a compromise is made among the various optimum conditions. The compromise is often complicated by the desire to cover an extensive wave-length range and still maintain nearly ideal intensity and resolution for the entire range.

The derivations of this article are intended to elucidate the properties of concave gratings at different angles, and to permit the evaluation of the imperfections of gratings under different conditions. On the basis of this information the best compromise for any particular use may then be chosen and such compromises have led to several different methods of mounting the concave grating. Of course, the ideal mounting of a grating would be a mechanism enabling its use at any angle of incidence and diffraction and thus allowing the best compromise for any particular problem at any accessible wave-length. Such a mounting requires that two of the three elements —grating, slit, and plateholder—be movable independently along the Rowland circle. This flexibility cannot be reconciled with the requirements for exact mechanical precision and rigidity in the relative position of the different parts,—at least not for instruments of ten or twenty foot size. In addition, it would require gratings of much higher perfection than hitherto ruled, and especially the elimination of the error of run. Therefore, all the mountings of grating spectro-

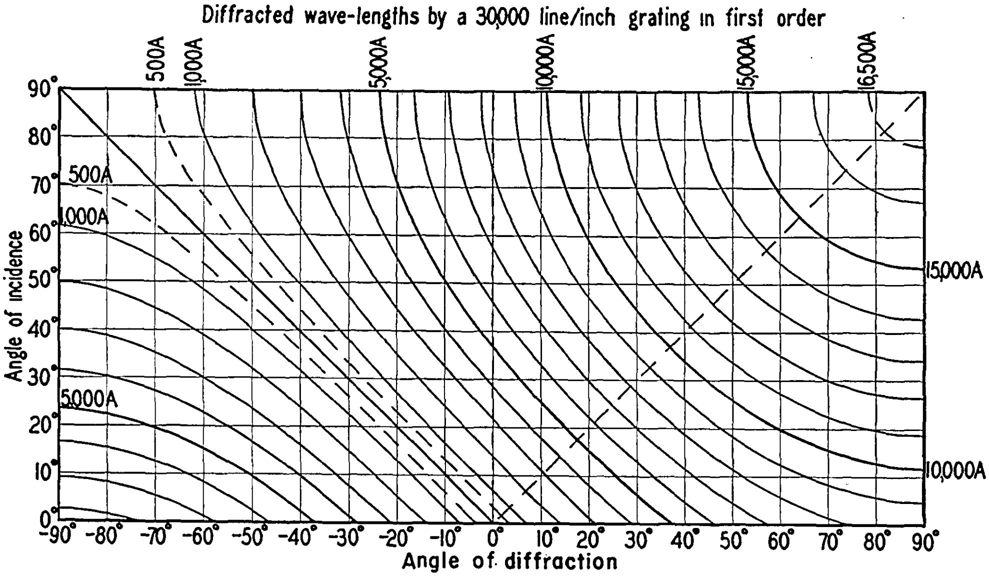


FIG. 2. Wave-lengths diffracted by a 12,000 lines per cm grating for all angles of incidence and diffraction.

graphs impose certain restrictions on the accessible range of angles.

Rowland, himself, used for image formation only the region near the normal of the grating: the angle $\beta$ was kept small,—between plus and minus ten degrees. For a 30,000 lines per inch grating in the first order, Fig. 2 shows that with Rowland's arrangement a range of about 2700A will be covered in one setting. The angle of incidence $\alpha$ can vary from about 20° to 80°. The range covered by the Rowland mounting is shown in Fig. 3. It may be remarked here that a mounting quite different from Rowland's in mechanical details, but essentially equivalent optically, was devised by Abney.[9]

A quite different mounting is that of Eagle.[10] This arrangement is similar to the Littrow mounting of the prism. The slit and plateholder are close together and fixed in relative position. To obtain the desired wave-length range, the grating is turned and moved back and forth, while the plateholder is also varied in tilt. The angles of incidence and diffraction are thus nearly equal, $\alpha \sim \beta$, and are varied simultaneously. In Fig. 3, the range covered by the Eagle mounting crosses the right-hand square diagonally. Slight variations of the Eagle mounting are possible, as the slit may be at one side or the other of the plateholder, or above it, $\alpha <, = \text{or} > \beta$. The relative advantages of these modifications will be discussed later.

A third type of mounting was devised by Paschen and Runge,[11] and is often referred to as the Paschen circle. A constant angle of incidence is chosen, often in the neighborhood of 45°, and the angle of diffraction extends over a very large range of the focal circle, perhaps from $-40°$ to $+70°$. In Fig. 3, this mounting is represented by a horizontal box.

It can be seen from Fig. 3 that the Eagle mounting covers the largest wave-length range of the three types of mountings, because it alters $\alpha$ and $\beta$ simultaneously and thus changes the function $(\sin\alpha + \sin\beta)$ by the largest amount. Of the standard mountings, the Eagle mounting, alone, makes accessible the longest wave-lengths, which are found in the upper right-hand corner of Fig. 2.

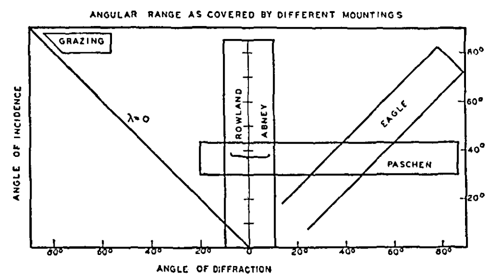


FIG. 3. Angular range covered by various grating mountings.

[9] W. deW. Abney, Phil. Trans. **177**, 457 (1886).
[10] A. Eagle, Astrophys. J. **31**, 120 (1910).
[11] C. Runge and F. Paschen, Anh. z.d.Abh. d. Berlin Akad. d. Wiss (1902).

For very short wave-lengths (<1000A) in the vacuum region, a mounting developed by Hoag[12] has been widely used.[13] This "grazing-incidence" mounting uses large angles of both incidence and diffraction—usually 80° or more for $\alpha$ and 60° and more for $\beta$. In Fig. 3, the wave-lengths handled by this mounting are found in the upper left-hand corner. It will be noticed from Fig. 2 that the dispersion in this region is very large and may be as much as 2.5 times larger than the dispersion when these wave-lengths are produced by small values of the angles $\alpha$ and $\beta$. The optical properties of this mounting are considerably different from those of the other mountings.

## V. THE FOCUSING CONDITIONS FOR THE ROWLAND CIRCLE AND FOR THE WADSWORTH'S MOUNTING

Returning to the discussion of Eq. (14), $m\lambda = d(\sin\alpha+\sin\beta)$, it is to be noted that this expression gives no information on the focal conditions for the concave grating, since $r$ and $r'$ do not appear. However, if $R$ has a finite value for the grating, the second and higher terms of the function $F$ have appreciable values, and their partial derivatives must be investigated and subjected to Fermat's condition.

Considering, first, the members, $F_2$ and $F_2'$, which have the largest values, and applying Fermat's condition, $\partial(F_2+F_2')/\partial w=0$, the introduction of the complete expressions (10b) and (11b) gives

$$\frac{\partial(F_2+F_2')}{\partial w}=w\left(\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}+\frac{\cos^2\beta}{r'}-\frac{\cos\beta}{R}\right)+\tfrac{3}{2}w^2\left[\frac{\sin\alpha}{r}\left(\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}\right)+\frac{\sin\beta}{r'}\left(\frac{\cos^2\beta}{r'}-\frac{\cos\beta}{R}\right)\right]+\cdots=0. \quad (16)$$

The first member of this expression vanishes if

$$\frac{\cos^2\alpha}{r}-\frac{\cos\alpha}{R}+\frac{\cos^2\beta}{r'}-\frac{\cos\beta}{R}=0, \quad (16a)$$

or

$$\cos\alpha\left(\frac{\cos\alpha}{r}-\frac{1}{R}\right)+\cos\beta\left(\frac{\cos\beta}{r'}-\frac{1}{R}\right)=0. \quad (16b)$$

One of the solutions of this equation is symmetrical in $\alpha$ and $\beta$, and is, by inspection,

$$r=R\cos\alpha;\quad r'=R\cos\beta. \quad (17)$$

These values make the second member of expression (16) equal to zero also.

These relations are the equations in polar coordinates of a circle of diameter $R$, on which the points $r$ and $r'$ lie. That is, the light source and spectral lines are on a circle, to which the grating is tangent, and which has as a diameter the radius of the grating blank. This circle is known as the "Rowland Circle."

Another solution of Eq. (16a) is not symmetrical and represents the case when parallel light strikes the grating, and the image is observed on the normal. Therefore

$$r=\infty \quad \text{and} \quad \beta=0°. \quad (18)$$

Introducing these conditions in Eq. (16a), there follows

$$r'=R/(1+\cos\alpha) \quad (19)$$

as the focal condition for points on the normal of the grating if parallel light strikes the grating at an angle $\alpha$. The focal curve given by Eq. (19) is a parabola (in polar coordinates), with its focus at the center of the grating.

The arrangement of the concave grating, described by Eqs. (18) and (19), is the so-called "stigmatic mounting" of Wadsworth.[14] The optical properties of this mounting and the mathematical treatment are widely different from those of the Rowland mounting. They will be discussed after the treatment of the Rowland circle.

## VI. THE OPTICAL IMAGE IMPERFECTIONS ON THE ROWLAND CIRCLE

### (a) The Conditions for the Cubic and Higher Order Terms to Vanish

Equations (14), (17), and (19) give all the necessary information for the calculation of the

[12] B. Hoag, Astrophys. J. **66**, 225 (1927).

[13] For instance by M. Siegbahn, B. Edlén, and J. Soederguist. Papers in Zeits. f. Physik (since 1930), J. E. Mack, P. G. Kruger, and I. S. Bowen in Phys. Rev. (since 1930).

[14] F. L. O. Wadsworth, Astrophys. J. **3**, 54 (1896).

dispersion of concave grating spectra. They do not, however, include the optical properties of the image formation or the aberrations of the image. To discuss these questions it is necessary to return again to Eqs. (15) and (16) and consider the higher members of the expressions, remembering that for the Rowland circle, with source and slit on the circle,

$$r = R\cos\alpha; \quad r' = R\cos\beta. \tag{17}$$

It will be seen that all the higher terms in the expressions (10b) and (11b) vanish, as well as the first terms, since all contain as factors one of the functions $(\cos^2\alpha/r - \cos\alpha/R)$ or $(\cos^2\beta/r' - \cos\beta/R)$, each of which becomes zero for the conditions of Eq. (17). Therefore, Eq. (16) is fulfilled exactly.

Furthermore, the same angular functions appear in every term in the expressions (10f) and (11f), as well as in their partial derivatives with respect to $w$, $\partial(F_6+F_6')/\partial w$. Again, then, the condition (17) for the Rowland circle reduces each term in these expressions to zero.

These results show the justification for the use of the Rowland circle with the concave grating. On the other hand, for any position of the light source, inside or outside of the Rowland circle, it is possible to satisfy the condition (16a)

$$\frac{\cos^2\alpha}{r} - \frac{\cos\alpha}{R} + \frac{\cos^2\beta}{r'} - \frac{\cos\beta}{R} = 0,$$

since for any given $r$ and $\alpha$, and with $\beta$ defined for the desired wave-length according to (14), a solution is

$$r' = \cos^2\beta \Big/ \left(\frac{\cos\alpha + \cos\beta}{R} - \frac{\cos^2\alpha}{r}\right). \tag{22}$$

For this solution, however, the two functions $(\cos^2\alpha/r - \cos\alpha/R)$ and $(\cos^2\beta/r' - \cos\beta/R)$ do not vanish separately. Therefore, the second and higher terms of the derivative $\partial(F_2+F_2')/\partial w$ do not vanish, since in them the two expressions have different coefficients,—functions of $\sin\alpha/r$ and of $\sin\beta/r'$ respectively. The Rayleigh condition (6d) must then be applied, and gives, for the second terms of the expression, (10b) and (11b)

$$\int_{-W/2}^{+W/2} \frac{\partial(F_2^{(2)}+F_2'^{(2)})}{\partial w} dw = \frac{W^3}{2}\left[\frac{\sin\alpha}{r}\left(\frac{\cos^2\alpha}{r} - \frac{\cos\alpha}{R}\right) + \frac{\sin\beta}{r'}\left(\frac{\cos^2\beta}{r'} - \frac{\cos\beta}{R}\right)\right] < \frac{\lambda}{4}. \tag{23}$$

Since all other quantities in Eq. (23) are fixed by the choice of $\alpha$, $r$ and $R$ and the resulting determination of $\beta$ and $r'$ from Eqs. (14) and (22), $W^3$ is the only variable and is therefore limited in size. For example, for a grating with a 20-foot focal length and 30,000 lines per inch, with $\alpha = 30°$ and $\beta = 0°$, giving 3000A in the second order, if the light source is placed, not on the Rowland circle, but at a distance, from the grating, $r_\alpha$, which is 10 percent greater, application of Eq. (23) shows that $W$ is limited to 5 cm. In other words, only 60,000 lines can contribute to the intensity of the spectral lines, and the resolving power in the second order is thereby restricted to about 120,000. If the deviation from the Rowland circle is 1 percent, the limiting value for $W$ is 10.3 cm, about 35 percent of the value for exact position on the circle, as shown later (Eq. (29)). These values, 5 and 10 cm, consider only the limiting influence of the second members of (10b) and (11b). Actually if $r \neq R\cos\alpha$, additional limitations arise from other terms in expressions (10) and (11),—especially from the first members of (10f) and (11f),—and reduce the useful grating width still further, by about 25 percent.[15]

It is clear that the Rowland circle is important as a locus for which the aberration of the grating arising from $\partial F/\partial w$ is reduced to a minimum. This aberration cannot be eliminated completely by the use of the Rowland circle mounting, however, because there are higher terms in $F$ which contain other angular functions that do not vanish when $r = R\cos\alpha$ and $r' = R\cos\beta$.

There are, of course, other imperfections in the image formation than those involving $\partial F/\partial w$, and some of these will be discussed later. Thus the expressions $F_3$ and $F_3'$ contain $l$, $z$, and $z'$, but

[15] Compare the results obtained by an approximated treatment of the same problem by D. L. MacAdam, J. Opt. Soc. Am. 23, 178 (1933).

not $w$. They therefore make no contribution to the above-mentioned aberration in the $x$–$y$ plane, and their influence can always be minimized by keeping $z$ and $l$ small. The optical imperfection arising from $\partial(F_3+F_3')/\partial l \neq 0$ is called astigmatism and will be discussed later.

### (b) The Aberration on the Rowland Circle

The largest contributions to the aberration, which is the analog of the spherical aberration of lenses, arise from the members $F_5+F_5'$, in which the expression $(w^2+l^2)^2$ is combined with an angular function $\phi$, defined from

$$F_5+F_5' = \frac{(w^2+l^2)^2}{8R^2}\left(\frac{1}{r}-\frac{1}{R}\cos\alpha + \frac{1}{r'}-\frac{1}{R}\cos\beta\right) = \frac{(w^2+l^2)^2}{8R^2}\cdot\phi. \quad (24)$$

For the Rowland circle, introducing (17), $\phi$ has the form

$$\phi = \frac{1}{R\cos\alpha} - \frac{\cos\alpha}{R} + \frac{1}{R\cos\beta} - \frac{\cos\beta}{R} = \frac{1}{R}\left(\frac{\sin^2\alpha}{\cos\alpha}+\frac{\sin^2\beta}{\cos\beta}\right). \quad (25)$$

This expression vanishes only if both $\alpha$ and $\beta$ are zero. It is symmetrical in $\alpha$ and $\beta$, becoming larger as both increase.

Since Eq. (24) is symmetrical in $w$ and $l$, it is convenient to put $\rho^2 = w^2+l^2$. $\rho$ then has the significance of the radius of a circle, the center of which is the midpoint of the grating, and the plane of which is tangent to the grating. The condition for image formation is that $F_5+F_5'$ is not greater than $\lambda/4$, as $\rho$ increases from zero to its limiting value. Therefore,

$$F_5+F_5' = \frac{\rho^4}{8R^3}\left(\frac{\sin^2\alpha}{\cos\alpha}+\frac{\sin^2\beta}{\cos\beta}\right) \leqslant \frac{\lambda}{4}. \quad (26)$$

Eliminating $\lambda$ by Eq. (14),

$$\frac{\rho^4}{8R^3}\left(\frac{\sin^2\alpha}{\cos\alpha}+\frac{\sin^2\beta}{\cos\beta}\right) \leqslant \frac{d}{4m}(\sin\alpha+\sin\beta), \quad (27)$$

or

$$\rho^4 \leqslant \frac{2R^3d}{m}\,\frac{\sin\alpha+\sin\beta}{\dfrac{\sin^2\alpha}{\cos\alpha}+\dfrac{\sin^2\beta}{\cos\beta}} \leqslant \frac{2R^3d}{m}\,\frac{2(\sin\alpha+\sin\beta)\cos\alpha\cos\beta}{(\cos\alpha+\cos\beta)(1-\cos\alpha\cos\beta)}, \quad (28)$$

and

$$\rho \leqslant \left(\frac{2R^3d}{m}\tan\frac{\alpha+\beta}{2}\cdot\frac{\cos\alpha\cos\beta}{1-\cos\alpha\cos\beta}\right)^{\frac{1}{4}}. \quad (29)$$

The expression gives the optimum aperture of a grating. It is a circle, of which the radius increases nearly linearly with the radius of curvature of the grating (more exactly: proportionally to $R_x^{\frac{3}{4}}$). The aperture is larger for coarser rulings (larger $d$) and smaller for higher order spectra (larger $m$).

Special cases of Eq. (29) have been given by other authors. Runge calculated for the Rowland mounting, with $\beta$ near zero, the optimum width, $B$, corresponding to $2\rho$

$$B = 2R\left(\frac{2d}{Rm}\cot\alpha\right)^{\frac{1}{4}}. \quad (29a)$$

Runge's expression follows from (29) by putting $\beta=0$, and writing $(1-\cos\alpha) = 2\sin^2(\alpha/2)$, and $2\sin(\alpha/2)\cdot\cos(\alpha/2) = \sin\alpha$.

Equation (29a) also holds for the Eagle mounting, because, replacing $\beta$ by $\alpha$, since $\alpha$ equals $\beta$ in the Eagle mounting, Eq. (29) reduces to (29a), as can be seen easily.

For grazing incidence, the optimum width is of importance, because it is much smaller than for the ordinary use of the grating. Mack, Stehn, and Edlén[16] derived for this case

$$B = 2.36\left(\frac{4\lambda R^3}{\pi}\cdot\frac{\cos\alpha\cos\beta}{(1-\cos\alpha\cos\beta)(\cos\alpha+\cos\beta)}\right)^{\frac{1}{4}}. \quad (29b)$$

This expression may be derived from (29) or better from (26), which contains $\lambda$. From (27) and (28)

$$\frac{\rho^4}{8R^3}\left(\frac{\sin^2\alpha}{\cos\alpha}+\frac{\sin^2\beta}{\cos\beta}\right) \leqslant \frac{\lambda}{4}, \quad (30)$$

[16] J. E. Mack, J. R. Stehn, and B. Edlén, J. Opt. Soc. Am. 22, 245 (1932).

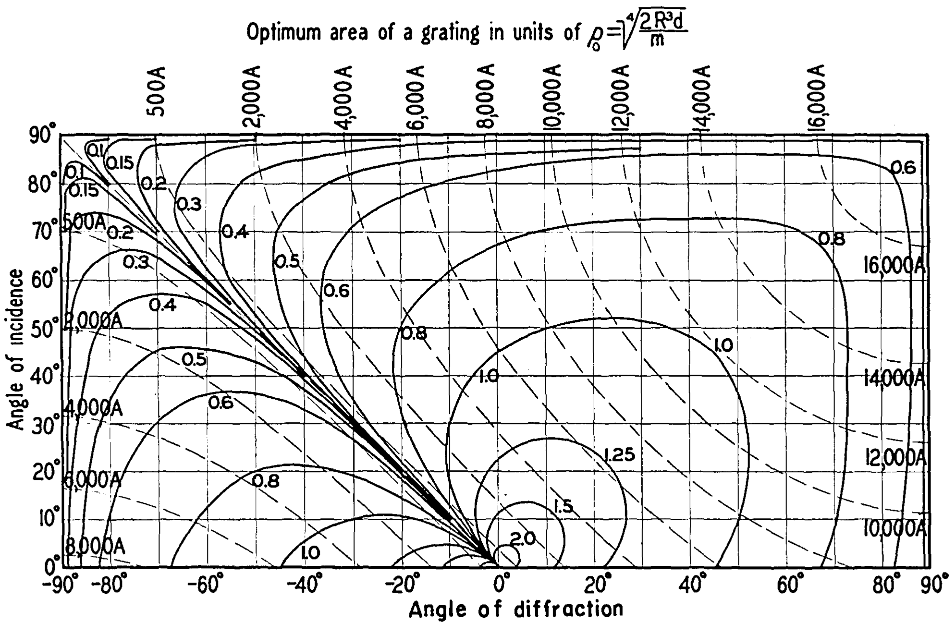


FIG. 4. Optimum radius of grating for all angles of incidence and diffraction in units of $\rho=(2R^3d/m)^{\frac{1}{4}}$.

and

$$\rho^4 \leqslant 2\lambda R^3 \Big/ \left(\frac{\sin^2\alpha}{\cos\alpha}+\frac{\sin^2\beta}{\cos\beta}\right)$$

$$=2\lambda R^3\left(\frac{\cos\alpha\cos\beta}{\sin^2\alpha\cos\beta+\sin^2\beta\cos\alpha}\right), \quad (31)$$

$$\rho=1.19\left(\frac{\lambda R^3\cos\alpha\cos\beta}{(1-\cos\alpha\cos\beta)(\cos\alpha+\cos\beta)}\right)^{\frac{1}{4}}.$$

But since $\beta=2\rho$, this gives (29b), with the numerical constant 2.38 (or $2.36(4/\pi)^{\frac{1}{4}}$) instead of 2.51. The small discrepancy of 5 percent arises from the somewhat different assumptions and methods of evaluation. MacAdam[17] has pointed out that the two assumptions are nearly equivalent.

In Fig. 4 the angular function

$$Y=\left(\tan\frac{\alpha+\beta}{2}\cdot\frac{\cos\alpha\cdot\cos\beta}{1-\cos\alpha\cdot\cos\beta}\right)^{\frac{1}{4}}, \quad (32)$$

that limits the circular opening of the grating is represented, the angle of incidence and of diffraction being the coordinates. The curves connect equal indicated values of $Y$, and the numerical amounts are given beside each curve. These numbers must be multiplied by the factor $(2R^3d/m)^{\frac{1}{4}}$ to obtain the radius $\rho$ in cm for the optimum aperture of the grating. For a survey, the following table gives the values of the fourth root expression for the gratings most frequently used:

TABLE I. Values of $(2R^3d/m)^{\frac{1}{4}}$, for $m=1$.

| $R$ | 1-meter | 10-foot | 21-foot | 30-foot |
|---|---|---|---|---|
| $d$ = 15000 lines/inch | 4.27 cm | 9.74 cm | 17.19 cm | 22.46 cm |
| $d$ = 30000 lines/inch | 3.59 cm | 8.19 cm | 14.46 cm | 18.89 cm |

For the most frequently used angles, the value of $Y$ is near to unity; then in the first order, the values in the table, multiplied by two, give the allowed width ($=2\rho$) of the grating. The restriction of the opening to a circle is not important for large values of $2\rho$, because the rulings are not long enough to cover even the allowed circle. But for grazing incidence, where $Y$ decreases to 0.2 and less, the circular diaphragm should be introduced.

### (c) Limitation of the Attainable Resolving Power and the Intensity Imposed by the Aberration

Whenever gratings are to be used to give a maximum resolving power in some broad wave-

[17] D. L. MacAdam, J. Opt. Soc. Am. 23, 178 (1933).

length range,[18] it is important to investigate how the choice of $R$ and $d$ of the grating can minimize the restrictions imposed by the aberration.

It is evident from Eq. (29) that, the variables $m$ and $d$ being constant, an increase of the radius of curvature increases the maximum resolving power only moderately, namely proportional to $R^{\frac{3}{4}}$, since the allowed width grows with this power. In the same grating ($R$ and $d$ being constant) the resolving power increases with higher order of the spectrum. The usual rule, that the resolving power is directly proportional to the order of the spectrum, holds only as long as the used surface of the grating is smaller than the area given by Eq. (29) for the highest order under consideration. If, however, the grating surface is larger than the area determined by Eq. (29) even for the first order, then $2\rho$ and therefore the number of lines contributing to the diffraction intensity are proportional to the fourth root of $1/m$. Therefore the resolution obtainable does not increase as 1, 2, 3, 4$\cdots$ but according to $m(1/m)^{\frac{1}{4}}=m^{\frac{3}{4}}$. In the very short wave-length region high orders of the lines appear frequently and are used for wave-length determination by comparison with standards of longer wave-length in low orders which they overlap. This is at present the principal method for determination of wave-length in the far ultraviolet, and for this application, the attainable resolution in the high orders, may be tabulated as follows:

| Order: | $m=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 . |
|---|---|---|---|---|---|---|---|---|---|---|
| Resolving power in units of first order | 1 | 1.68 | 2.28 | 2.83 | 3.35 | 3.83 | 4.30 | 4.75 | 5.19 | 5.62 |

In higher orders, the surface of the grating has to be limited to the optimum, or the resolution would be much lower.[19]

From the graph in Fig. 4 it is apparent that $Y$ for any given wave-length has a maximum for $\alpha=\beta$, the optical arrangement of the Eagle mounting. For this mounting, as noted above, Eq. (29a), the condition for optimum width is, $\rho\leq(2R^3d/m\cdot\tan\alpha)^{\frac{1}{4}}$ (for $\alpha\sim\beta$). For short wave-lengths, in this case $\lambda<4000$, $\tan\alpha$ can be replaced in Eq. (29a) by $\sin\alpha$. By introducing Eq. (14) in the form: $\sin\alpha=\lambda/2\cdot m/d$ (since $\alpha=\beta$), the expression becomes,

$$\rho\leq\left(\frac{4R^3d^2}{m^2\lambda}\right)^{\frac{1}{4}} \text{ (for } \lambda<4000\text{A, } \alpha\sim\beta). \quad (33)$$

This equation is useful in the choice of the spacing of the grooves for a grating when a definite radius of curvature has to be used, and a certain range of (short) wave-lengths in a certain order is to be investigated. It is found from Eq. (33) that the allowed radius is proportional to $(d)^{\frac{1}{2}}$. The resolving power $\lambda/d\lambda$ of any grating is given by the number of lines $N=2\rho/d$, and therefore the maximum resolution is determined by:

$$\frac{\lambda}{d\lambda}=\frac{2\rho}{d}\leq2\left(\frac{1}{d}\right)^{\frac{1}{2}}\cdot\left(\frac{4R^3}{m^2\lambda}\right)^{\frac{1}{4}},$$
$$\text{for } \lambda<4000\text{A}; \alpha\sim\beta<15°. \quad (34)$$

It follows that the maximum resolving power is proportional to the square root of the number of lines ruled per cm, other things being equal. An increase of $N$/cm by the factor of two enhances the attainable resolving power only by the factor 1.41 for short wave-length and at nearly normal incidence, whereas the dispersion is increased by the factor two, provided that the grating used is always large enough to reach to the limits set by the aberration. The same result holds approximately for grazing incidence, but here in addition another factor favoring the resolution results from the fact that higher orders of any particular wave-length appear under smaller angles, that is at "less grazing" incidence.

It may be mentioned that the intensity of the light coming from the grating is also limited by the aberration, since the exposed surface is $\rho^2\pi$. The increase of the number of lines by the factor two therefore reduces the intensity by the factor two, everything else being equal according to Eq. (34). But there are many other factors

[18] For a small wave-length range, the stigmatic mounting (with the spectrum on the normal) is preferable.

[19] See the photograph in the paper of Mack, Stehn, and Edlén (reference 16).

entering intensity questions. For very short wave-length and nearly normal incidence, the parts of the surface between the grooves contribute most to the diffracted image. The increase of the number of lines goes on largely at expense of the space between the grooves, and therefore the intensity will decrease faster than expected from Eq. (34).

## VII. ASTIGMATISM OF THE CONCAVE GRATING (GENERAL)

One of the outstanding optical properties of the concave grating, distinguishing it from all the other optical instruments containing lenses or spherical mirrors, is the fact that light can be utilized which is incident at large angles of incidence. A necessary result of these large angles is a set of imperfections in the image formation, of which the most striking one is astigmatism. The amount of astigmatism present at large angles is enormous. It can be tolerated only because a slit is used as light source with parallel lines as images, and because the grating is set with its grooves parallel to the slit; thus the astigmatism is parallel to the lines. At right angles to the astigmatism a perfect focus can be maintained along the Rowland circle.

Slit sources with line images in monochromatic light were used before the concave grating was invented. Rowland did not effect any improvement in the astigmatism, but he obtained nearly the same optical performance given by prism spectrographs. The astigmatism, however reduces the intensity of the spectra considerably. This loss was not too serious in Rowland's time because on the one hand the gratings were never bright in comparison to prisms, and on the other hand there were, in addition to the sun, enough very bright spectra obtainable and not yet investigated in high dispersion. A point light source on the slit is drawn out to a spectral line as image. For some purposes this effect is even an advantage. Rowland wrote: "Indeed it adds to the beauty of the spectra, as the horizontal lines due to dust in the slit are never present, as the dust has a different focal length from the lines of the spectrum."

During the last fifty years, however, the bright spectra have been thoroughly investigated, while the increase in resolving power of the gratings requires narrower slits and finer grained plates, so that the loss in intensity by the astigmatism becomes a serious problem. Furthermore, the modern development of spectroscopy often utilizes the length direction of the spectral lines for introducing another variable along this coordinate. The commonest cases are the application of intensity marks by step wedges or sectors for quantitative spectrochemical analysis, or the interference pattern obtained by Lummer plates or Fabry-Perot interferometers for investigation of the hyperfine structure of spectral lines. All these types of spectroscopic problems are not feasible with an instrument which does not give a stigmatic image of the slit. For these applications prism spectrographs are still built and in demand. Several methods have been tried of eliminating the astigmatism. One arrangement mounts the concave grating stigmatically, but it limits the useful aperture of the grating largely to the region near its normal, and thus it limits the spectral range accessible in one setting. Other devices apply cylindrical lenses; their usefulness is limited to narrow spectral ranges too, or other adjustments have to be performed. Some devices simply try to reduce the astigmatism and to minimize the intensity loss, but without giving a truly stigmatic image.

In general, it may be stated that the astigmatism is inherent in the concave mirror, which is the basis of the grating, as soon as the angle of incidence or diffraction is other than zero and that therefore no radical removal of it is possible, but only a compensation by auxiliary optical elements.

As stated above, a point light source at the slit is imaged as a vertical spectral line, if the grating grooves are vertical. On the Rowland circle, there is no focusing along the vertical direction. But outside the circle, the point source forms a horizontal line which corresponds to a focus in the vertical direction and is called the "vertical focus."

### (a) Quantitative Treatment of Astigmatism on the Rowland Circle

The directions of the light paths in the vertical planes that include $z$, $z'$, and $l$ at the angles $\alpha$ and $\beta$ to the normal, are governed by the members $F_3+F_3'$ of our formulas (10, 11) in first approxi-

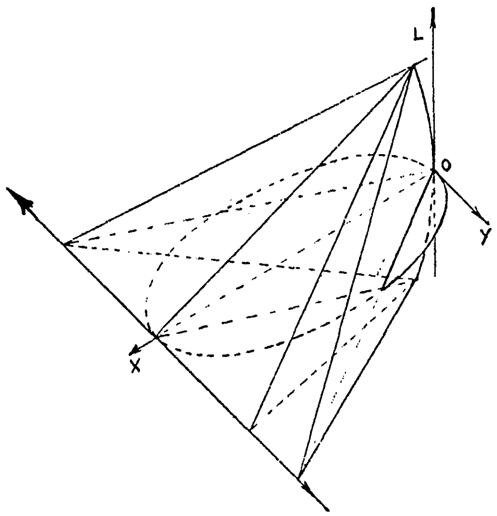


FIG. 5. Image formation for rays inclined to the Rowland plane.

mation, whence:

$$F_3+F_3'=\tfrac{1}{2}l^2\left(\frac{1}{r}-\frac{\cos\alpha}{R}+\frac{1}{r'}-\frac{\cos\beta}{R}\right)$$
$$-l\left(\frac{z}{r}+\frac{z'}{r'}\right)+\frac{z^2}{2r}+\frac{z'^2}{2r}. \quad (35)$$

The point light source at $z$, $\alpha$, $r$ can only be focused at some point $z'$, $\beta$, $r'$, if Fermat's theorem (6b) is fulfilled for the vertical coordinate $l$ of the grating. Therefore:

$$\frac{\partial(F_3+F_3')}{\partial l}=l\left(\frac{1}{r}-\frac{\cos\alpha}{R}+\frac{1}{r'}-\frac{\cos\beta}{R}\right)$$
$$-\frac{z}{r}-\frac{z'}{r'}=0. \quad (36)$$

If any point $z$ is to be focused at some point $z'$, it is to be expected that if $z=0$, the focus will also lie in the equatorial plane, that is $z'=0$. Upon introducing this relation, Eq. (36) becomes:

$$\frac{\partial(F_3+F_3')}{\partial l}=l\left(\frac{1}{r}-\frac{\cos\alpha}{R}+\frac{1}{r'}-\frac{\cos\beta}{R}\right)=0. \quad (37)$$

Since $l$ has a finite value, the expression in the parenthesis must become zero, if a focus exists at all. The locus of the "secondary" stigmatic foci is therefore:

$$1/r-\cos\alpha/R+1/r'-\cos\beta/R=0. \quad (38)$$

This condition is fulfilled symmetrically for the condition:

$$r=R/\cos\alpha;\quad r'=R/\cos\beta. \quad (39)$$

The curve representing this equation is a straight line, tangent to the Rowland circle at $\alpha=\beta=0$, or through the center of curvature of the blank. From this, some conclusions can be derived that have already been pointed out by Sirks:[20]

1. If the slit lies at the normal of the grating and on the Rowland circle ($\alpha=0$, $r=R$), then points in the slit are not focused as points on the Rowland circle; they are brought to a horizontal astigmatic focal line outside of the Rowland circle. The locus of these foci is the tangent to the Rowland circle at its intersection with the normal of the grating.

2. For any wave-length appearing at the normal, horizontal crosshairs in the incident beam will be focused sharply if they lie on the tangent to the Rowland circle at the normal.[21]

The focal properties of the grating for rays that are inclined to the horizontal Rowland plane can be visualized as follows: points lying on the tangent to the Rowland circle through the normal are imaged into other points of the same tangent. In Fig. 5, this behavior is represented. The vertical direction of the grating extends along $L$. The light beams from focus to focus lie inside a tetrahedron, one edge of which is a vertical groove of the grating, and the other edge the focal line, the "normal tangent" of the Rowland circle. For comparison, the focusing of points on the Rowland circle is represented in Fig. 6. Rays diverging from points on the Rowland circle, that are reflected from points along the horizontal extension $w$ of the grating, form images at other points of the Rowland circle. The family of the light beams from focus to focus forms a circular horizontal disk. All the points of the circle in Fig. 6 lie inside the straight line of Fig. 5. The

[20] J. L. Sirks, Astron. and Astrophys. **13**, 763 (1894).

[21] It may be emphasized, that these conditions are only fulfilled for the spectral lines appearing at the normal. Sirks treated only this case, because at that time no other than the Rowland mounting was known. For spectral lines appearing far away from the normal, the construction of a tangent does not lead to the outside focus for horizontal lines, as is sometimes erroneously stated. In Fig. 8, the distances of those foci outside of the slit are represented in a diagram.

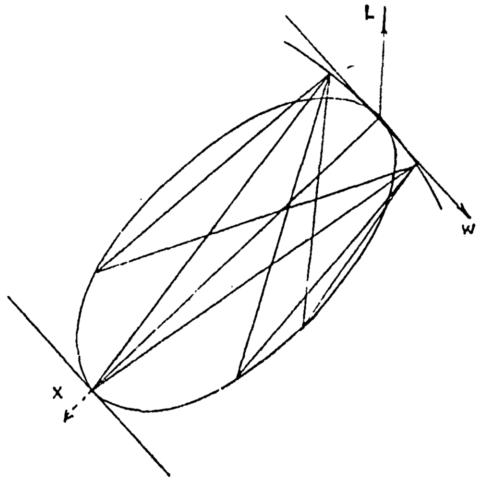


FIG. 6. Image formation in the Rowland plane.

conception of astigmatism refers to the fact that there are two distinct focal curves. Since a general quantitative measure for the astigmatism has not been established, the difference of the two focal points at any given angle or the ratio of these lengths or any other well defined property may be used for its measurement.

The comparison of Fig. 5 with Fig. 6 makes it evident that on the Rowland circle a stigmatic image is formed only if the source and the image are located at the normal. The astigmatism remains negligible for points near to the normal, that is for $\alpha$ and $\beta$ smaller than about 3°. Using gratings with 15,000 lines/inch, this region contains the wave-lengths up to 2000A.

Equations (38) and (39) do not contain $\lambda$ or $d$; the astigmatism is a function only of the angles of incidence and diffraction.

Equation (38) can also be fulfilled asymmetrically by the condition:

$$r=\infty,\quad \beta=0°. \tag{40}$$

This property has been utilized by Wadsworth in his so-called stigmatic mounting which is treated later (Section XI).

The amount of astigmatism at any Rowland circle mounting may be very considerable, depending on $\alpha$ and $\beta$. For a point source at the circle, the conjugate vertical focus of some wave-length may fall at infinity, and it is even possible that divergent light with respect to $l$ emerges from the grating. We can study this relation by introducing into Eq. (38) the condition that the source lies on the circle $r=R\cos\alpha$, which gives:

$$\frac{1}{R\cos\alpha}-\frac{\cos\alpha}{R}+\frac{1}{r'}-\frac{\cos\beta}{R}=0. \tag{41}$$

The condition for emergence of parallel light is that $r'$ becomes infinite, and therefore $1/r'=0$; the equation reduces to:

$$\frac{1}{R\cos\alpha}-\frac{\cos\alpha}{R}=\frac{\cos\beta}{R},$$

or

$$\cos\beta=\frac{\sin^2\alpha}{\cos\alpha}=\sin\alpha\cdot\tan\alpha. \tag{42}$$

This means that for a considerable range of $\alpha$, parallel light will emerge at a certain angle $\beta$. For small angles of incidence $\alpha$, the light will be parallel at large angles $\beta$; for larger angles $\alpha$, $\beta$ for parallel light decreases rapidly. For $\alpha=45°$, parallel light emerges at $\beta=45°$, and for $\alpha=52°$, the light emerging normal to the grating is parallel.

The light emerging from the grating is convergent if $r'$ is positive; it is divergent if $r'$ becomes negative. By replacing $1/r'$, in Eq. (41), by small positive or negative values the following additional relations are obtained:

if $\qquad\cos\beta>\sin\alpha\cdot\tan\alpha,$

the emergent light is convergent in vertical plane;

if $\qquad\cos\beta=\sin\alpha\cdot\tan\alpha,$

the emergent light is parallel in vertical plane;

if $\qquad\cos\beta<\sin\alpha\cdot\tan\alpha,$

the emergent light is divergent in vertical plane.

### (b) Quantitative Evaluation of the Magnitude of Astigmatism

In order to calculate the quantitative influence of the astigmatism on the image formation at the Rowland circle, the values $R\cos\alpha$ and $R\cos\beta$ (Eq. (17)) for $r$ and $r'$ are introduced into Eq. (36). Thence:

$$l\left(\frac{1}{R\cdot\cos\alpha}-\frac{\cos\alpha}{R}+\frac{1}{R\cdot\cos\beta}-\frac{\cos\beta}{R}\right)-\frac{z}{R\cdot\cos\alpha}-\frac{z'}{R\cdot\cos\beta}=0. \tag{43}$$

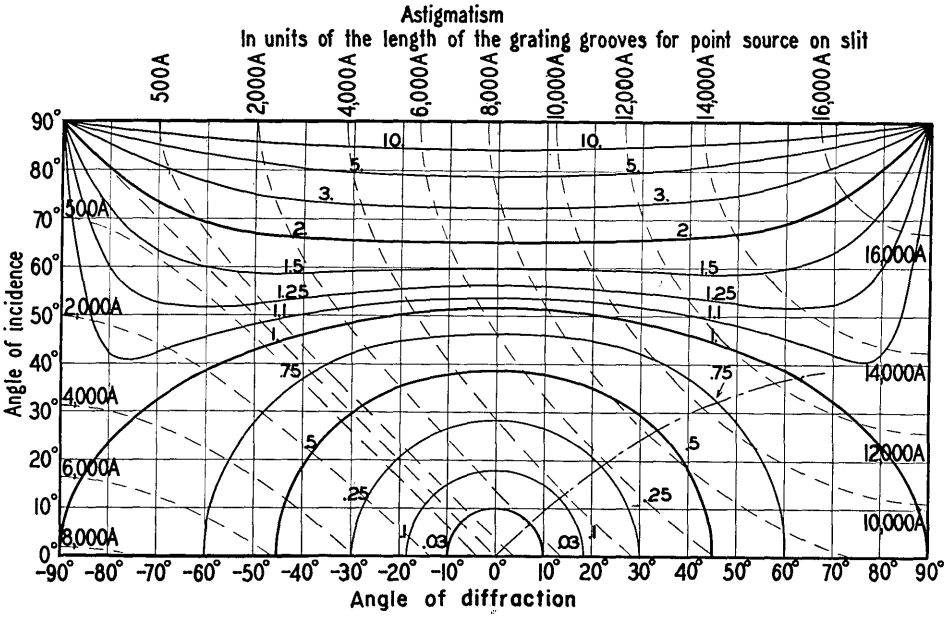


FIG. 7. Astigmatism in units of the length of the grating grooves for a point source at the slit for all angles of incidence and diffraction.

By transformation:

$$l\left(\frac{\sin^2\alpha}{\cos\alpha}+\frac{\sin^2\beta}{\cos\beta}\right)-\frac{z}{\cos\alpha}-\frac{z'}{\cos\beta}=0. \qquad (44)$$

This equation connects the lengths of source and of image with the lengths of the rulings. It should be noted that the equation does not contain $R$ in any power; the astigmatism is therefore independent of the size of the Rowland circle mounting; it has already been shown to be independent of the grating constant and of the order of the spectrum. The length of the images that are formed by a point light source at the slit may be found by setting $z=0$ in Eq. (44). Whence:

$$z'=l\left(\sin^2\beta+\sin^2\alpha\cdot\frac{\cos\beta}{\cos\alpha}\right). \qquad (45)$$

The numerical evaluation of this formula for some particular angles $\alpha$ has been published by Dieke.[22] A knowledge of the numerical values is frequently wanted, especially when a new grating is to be mounted, for which the optimum angle of incidence for some purpose is to be chosen. Therefore the complete set has been calculated for all the possible values of $\alpha$ and $\beta$. The results are represented in Fig. 7, where the coordinates are again the angles of incidence and diffraction. The curves connect the points for which the angular function $\Gamma$,

$$\Gamma=\sin^2\beta+\sin^2\alpha\cdot(\cos\beta/\cos\alpha)=z'/l, \qquad (46)$$

has the same value; they could be called "iso-astigmats." The numerical values of $\Gamma$ are designated; they mean therefore the ratios $z'/l$ for a point light source, i.e., the lengths $z'$ of its images measured in the unit of length $l$ of the grooves of the grating. The curve 1.0 represents the emergence of parallel light, which has been discussed above qualitatively (Eq. (42)). Inside this curve convergent light emerges, and outside of it, divergent light. The dashed curves indicate the wave-lengths for a grating with 30,000 lines/inch, as in Fig. 2.

The superposition of these two sets of curves allows one to select the arrangement of slit and plate for any desired wave-lengths—which corresponds to a minimum of astigmatism. The curve in Fig. 7 drawn by points and dashes —·—·—· represents the angles to be chosen if a minimum of astigmatism for a particular wave-length is desired. It will be seen that the Eagle mounting ($\alpha\sim\beta$) covers the range of minimum astigmatism —a fact already stressed by Eagle[23]—for angles

[22] G. H. Dieke, J. Opt. Soc. Am. **23**, 274 (1933).

[23] A. Eagle, Astrophys. J. **31**, 120 (1910).

up to approximately 35°. For longer wave-lengths, whose numerical values depend on the grating constant $d$, it is advantageous to work at a constant angle of incidence of about 35°, until the angle of diffraction $\beta$ becomes very large. Values for $\beta$ larger than about 75° are unfavorable[24] because the adjustment becomes more difficult and the coma becomes too large. It is then better to increase the angle $\alpha$ again to reach the range of longer waves.

The astigmatism for the longest wave-lengths accessible by a given grating is smaller the nearer $\beta$ is chosen to 90°. If an Eagle mounting is constructed to be used in the range of long waves, or high orders of short waves, and small astigmatism is wanted, the angle of incidence should be chosen smaller than the angle of diffraction, i.e., the slit should be beyond that edge of the plateholder which is nearer to the normal of the grating. (It will be shown that the attainment of high resolving power favors the opposite arrangement of slit and plate.) (See part c of Section VIII.)

Comparison with Fig. 2 shows that the Rowland and the Abney mounting of the concave grating cover ranges of the angles which are quite unfavorable with respect to astigmatism. With a 30,000 lines/inch grating, at 8000A (=2nd-order 4000A), the spectral lines are already three times as long as the grooves of the grating, and the intensity is thus considerably reduced. It has been proposed to use a very long slit to compensate for this loss, but as discussed in the section on coma the spectral lines will be imaged very poorly in this case. With the Runge-Paschen mounting the astigmatism may be kept within tolerable limits. The usual arrangement with an angle of incidence about 45° produces spectral lines of nearly uniform astigmatism ($\Gamma=0.72$ to 1.2) all around the circle. It seems feasible to use a second entrance slit at $\alpha$ about 20° for work at medium wave-lengths, in order to utilize the smaller astigmatism in this region.

For the concave grating in grazing incidence a very large astigmatism is always present. But here, a compensation for the intensity loss occurs: at grazing incidence, the reflectivity of polished surfaces for short waves is much larger than at normal incidence. Furthermore, the higher dispersion obtainable by this mounting is generally the most important factor in the range of very short waves.

### (c) The Secondary Focal Curves

From the general Eq. (38) it was shown that any point on the tangent to the normal of the Rowland circle is focused "vertically" at some other point of that tangent. It is also important to know where point light sources must be located in order to give a point image in "vertical focus" on the Rowland circle.

The locus $r_v$ for such light sources is obtained by introducing $r'=R\cdot\cos\beta$ into Eq. (38).

$$\frac{1}{r}=\frac{\cos\alpha}{R}+\frac{1}{R\cdot\cos\beta}-\frac{\cos\beta}{R}=0; \tag{47}$$

$$\frac{1}{r_v}=\frac{\cos\alpha+\cos\beta}{R}-\frac{1}{R\cdot\cos\beta};$$

$$r_v=R\cdot\frac{1}{\cos\alpha-\sin\beta\cdot\tan\beta}. \tag{48}$$

For any angle of diffraction $\beta$ and incidence $\alpha$, there is a certain distance $r_v$ from a grating, at which a point light source must be located to be imaged as a horizontal line on photographic plates that are placed on the Rowland circle. If then the incident light passes through the slit, monochromatic images of the slit are obtained, but the length of the slit is reduced to a point on the plate.

In practice, the distance of the point light source from the grating is not so important as its distance from the slit, since that is the part of the grating spectrograph easily accessible from the outside. For calculating this distance, $s$, the distance between slit and grating is subtracted

[24] In some treatments the reason given for avoiding large angles is that then the angular aperture of the grating becomes very small. This is not the case however because the width of the grating can be approximated very well by a chord on the Rowland circle, and the angle subtended at any point of a circle by the two end points of a chord is strictly constant. This geometrical property can in fact be used for derivation of the Rowland circle as focal curve of a concave grating. On the other hand, the aperture with respect to the length $l$ of the grating gets much larger with increasing angles, as $1/\cos\alpha$ or $1/\cos\beta$ respectively.—This conclusion, however, holds only as long as the aperture of the grating is limited by its own size, not by the aberration. (See Section X.)

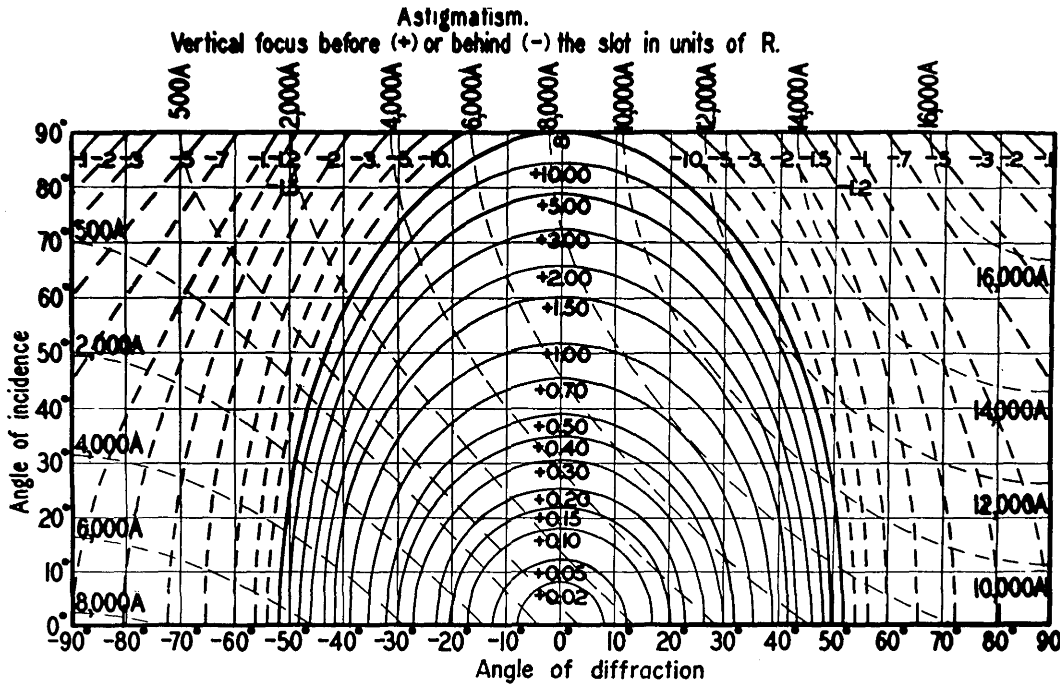


FIG. 8. Distance of vertical focus before (+) or behind (−) the slit in units of grating radius $R$ for all angles of incidence and diffraction.

from $r_v$, giving:

$$s = r_v - r = R\left(\frac{1}{\cos\alpha - \sin\beta\tan\beta} - \cos\alpha\right)$$
$$= R\cdot\Lambda. \quad (49)$$

The angular part of this function, $\Lambda = s/R$, has been plotted in a graph, represented in Fig. 8. In view of the practical importance of this function, the values are given in small steps. The curves connect the points with equal $\Lambda$, i.e., the values of the distances $s$ measured in units of the radius of curvature of the grating blank. Inspection shows that for the normal ($\alpha = \beta = 0$) $s = 0$, and that $s$ rises rapidly to unity for increasing angles, and to infinity for $\alpha = 45°$, $\beta = 45°$. In this region then, parallel light striking the slit and the grating will be focused as a point on the Rowland circle. Proceeding to larger angles, the grating must be illuminated with convergent light to give a focus (vertically) at the circle; the amount of convergence is given by a negative figure for $\Lambda$ which designates the position of the focus measured in units of $R$ from the slit towards the grating.

Of course it is not necessary to put the actual light source at the distance $\Lambda = s/R$; it is sufficient to create a virtual image there by means of a spherical or better a cylindro-convex or concave lens.

In reality, all light sources are of finite size, and therefore the grating will produce a certain magnification of the image. If the height of the light source (or its virtual image) in the outside focus is designated by $z_v$, then the height $z'$ of the spectral line at the Rowland circle will be given by simple geometry and by use of Eq. (44) as:

$$z' = z_v \cdot \frac{r'}{r_v} = z_v \cdot \cos\beta(\cos\alpha - \sin\beta\tan\beta) \quad (50)$$
$$= z_v(\cos\alpha\cdot\cos\beta - \sin^2\beta).$$

The value of the angular function is always smaller than unity (it is equal to unity for $\alpha = \beta = 0$). The consideration of Eq. (50) is necessary for the choice of a cylindrical lens, placed between the slit and the source, to correct the astigmatism. The decision depends of course on the size of the light source and on the angles $\alpha$

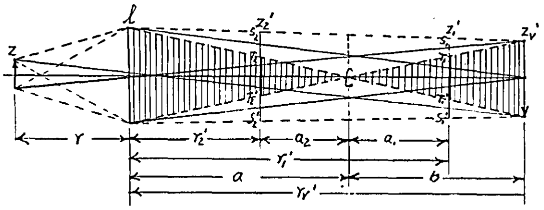

FIG. 9. Illumination of the image formed by a concave grating.

and $\beta$ which must be used to photograph the desired wave-lengths with the given apparatus.

Considerations of this kind may become more complicated, if the maximum resolving power is wanted. The occurrence of the coma limits the length of the slit and of the grating grooves that may be illuminated. These restrictions will be referred to later. (See Section VIII.)

### (d) The Intensity Distribution Along the Spectral Lines as a Result of the Astigmatism

If a light source of the length $z$ is used at the slit, Eq. (44) can be written as:

$$z' = l\left(\sin^2\beta + \sin^2\alpha\cdot\frac{\cos\beta}{\cos\alpha}\right) - z\cdot\frac{\cos\beta}{\cos\alpha}, \quad (51)$$

in order to represent the length of the image. $z$ and $z'$ have opposite signs, corresponding to the inversion of the slit image by the reflection at the grating. For $l$, the sign of the coordinates must be introduced too. The relations are somewhat complicated, but must be considered because it has been proposed to use the astigmatism for quantitative intensity measurements.[25] This procedure may be feasible, if the necessary precautions are taken.

The geometry of the vertical sections through the sheet of light that is defined by slit, grooves, and spectral line is represented in Fig. 9. The relations are the same as for a cylindrical lens, or, since only one plane section through the beams is considered, as for a spherical lens. It is assumed that convergent light emerges from the grating, represented by $l$ in Fig. 9.

The light beam originates in $z$ (now designating the full length of the slit) and is focused by $l$ at a distance $r_v'$ as a line of length $z_v'$. The peculiarity in this arrangement of the grating is that the light beam is not intercepted by a screen at the focal distance, but much nearer at the distance $r_1'$, i.e., at the Rowland circle. From Fig. 9 it can be seen that the light sheet between $l$ and $z_v'$ contains two triangular parts with their bases in $l$ and in $z_v'$ and a common vertex at $C$. All the points within these triangles receive light from all the points of $z$, whereas points lying in the remaining two triangles shown receive light only from parts of $z$. The light sheet emerging from $l$ can therefore be divided into two parts: a fully illuminated and a partly illuminated one, and a screen inserted at some distance $r_1'$, or $r_2'$ closer to $l$ than $r_v'$ will show these two parts distinctly. In Fig. 10 is given a graph of the intensity distribution along $z_1'$, which is chosen as abscissa. Over a central part $T_1T_1'$, there is a uniform maximum intensity, but this decreases linearly toward the outer parts of the sheet of light, until its boundaries are reached at $S_1$ and $S_1'$. The linearity of the decrease can be derived easily by construction of light beams from points along $z$.

It may be remarked here that this theoretical linearity can be used for the reduction of the density on the photographic plate, in the same manner that a wedge is used in photometric work for printing intensity marks. These calibrating marks may be used in comparing the intensity of not too distant spectral lines with each other.

The lengths of $T_1T_1'$ and $S_1S_1'$ can be readily calculated. For the general case, two possible positions of the screen at $r_1'$ and at $r_2'$ are considered; *viz.* inside and outside of the crossing point $C$, whose position depends not only on $r$ and $r_v'$ but also on the magnitudes of $z$ and $l$.

From simple geometry, the following proportions can be derived:

$$T_2T_2':\ l = a_2 : a;\ a : b = l\ : z_v';\ (S_1S_1' - z_v') : (b - a_1) = (l - z_v') : r_v';\ a_1 = r_1' - a$$
$$T_1T_1' : z_v' = a_1 : b;\ z : r = z_v' : r_v';\ (S_2S_2' - z_v') : (b - a_2) = (l - z_v') : r_v';\ a_2 = b - r_2'$$
$$a + b = r_v'$$

[25] G. H. Dieke, J. Opt. Soc. Am. **23**, 280 (1933); Sister M. I. Bresch, J. Opt. Soc. Am. **28**, 493 (1938).

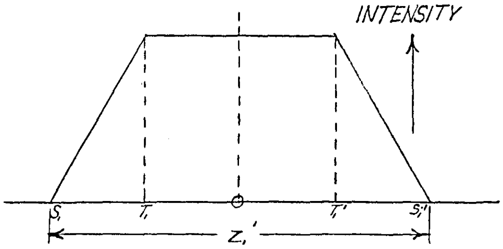


FIG. 10. Intensity distribution in grating line-image.

From the grating equations, by solving Eq. (38) for $r=R\cos\alpha$, there follows:

$$r_v'=\frac{R}{\cos\beta-\sin\alpha\cdot\tan\alpha};$$

$$r_1'=R\cdot\cos\beta_1;$$

$$r_2'=R\cdot\cos\beta_2.$$

By combination of these equations and substitution:

$$z_v'=z\cdot\frac{1}{\cos\alpha\cdot\cos\beta_{1,2}-\sin^2\alpha}$$

and

$$\frac{r_{1,2}'}{r_v'}=\cos\beta(\cos\beta-\sin\alpha\cdot\tan\alpha). \quad (52)$$

The final results yield expressions that show $T_1T_1'=T_2T_2'$ and $S_1S_1'=S_2S_2'$, so that omitting the indices:

$$TT'=l\left(\sin^2\beta+\sin^2\alpha\cdot\frac{\cos\beta}{\cos\alpha}\right)-z\frac{\cos\alpha}{\cos\beta};$$
$$SS'=l\left(\sin^2\beta+\sin^2\alpha\cdot\frac{\cos\beta}{\cos\alpha}\right)+z\cdot\frac{\cos\alpha}{\cos\beta}. \quad (53)$$

The value $TT'$ designates the central part of the astigmatic image, which has uniform intensity; the length $SS'$, however, designates the full lengths of the astigmatic image $z'$ at the Rowland circle. This function has been evaluated for $l=75$ mm and $z'=5$ mm. The values calculated for the necessary slit lengths are plotted in Fig. 11 with the coordinates $\alpha$ and $\beta$, and for easier orientation the wave-length "isochromats" with $\lambda$ values for first order of a 30,000 lines/inch grating have been plotted by dashed lines. In evaluation of this formula, Fig. 9 has been considered. The construction must be reversed to give $z$ for a given $r'$ and $z'$ ($TT'$ with maximum intensity$=5$ mm). It is found that two different slit lengths are possible according to the choice of $r'$ between the lens and the crossing point or outside of that distance. Two different slit lengths result, of which the shorter one has been used in plotting the graph.

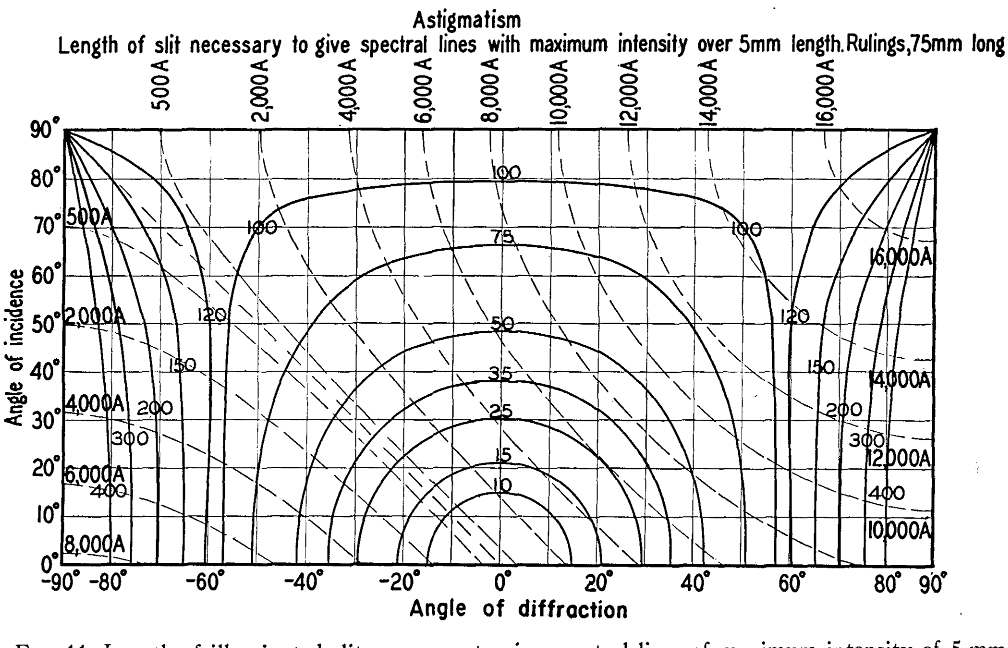


FIG. 11. Length of illuminated slit necessary to give spectral lines of maximum intensity of 5-mm length for all angles of incidence and diffraction of a grating with rulings 75 mm long.

It is of course not feasible to use slits of the extreme length that is given in millimeters on the graph. The coma then occurring is prohibitive, if even a moderate resolution is desired. There are other means for reducing the intensity loss from astigmatism by auxiliary optical devices.

### VIII. COMA

In the series development the members (10c) and (11c) contain $l$, but not $w$. They represent therefore an aberration that is restricted to the planes containing $z$ and $l$, or $z'$ and $l$, respectively. The spectral lines, therefore, remain strictly straight lines.

A simple geometrical consideration, however, shows that this statement is only an approximation. If any point of the slit is connected with the points along one groove of the grating, then this triangular beam of light will be a plane sheet only for the case $\alpha=0$; otherwise it will be curved. In fact, it will be part of a surface of an oblique circular cone. The same relations hold for the diffracted sheet of light, which will be plane only for $\beta=0$, and otherwise will be part of a surface of an oblique circular cone, the top of which lies on the secondary focal curve. A consideration of the astigmatic elongation produced on the Rowland cylinder by this system of curved sheets shows that a curvature of the spectral lines must appear. Furthermore, as soon as a slit longer than a point is used, a one-sided shading of the line will result which corresponds to the aberration which in lenses is designated as coma. This effect widens the spectral lines, even at $z'=0$, and damages the resolving power. The coma must therefore be restricted in its magnitude, and from simple geometry it is evident that a limitation of the length of the slit and of the rulings is required. The calculation of the permissible values from geometrical considerations is, however, rather cumbersome and difficult. By means of the series development of the generalized theory, however, these evaluations can be carried on readily with a good degree of approximation. The series development was in fact originally introduced for this calculation of coma and curvature, since the geometrical treatment was too laborious.

In the series development of the general formula (9), the coma and the curvature of the spectral lines are governed by the terms (10d) and (11d) that contain the cross products $w\cdot l^2$. The partial derivatives of (10d) and (11d) with respect to $l$ are not important, because they contribute only small deviations in the direction $z'$—they increase or decrease the already present astigmatism by a small fraction, yielding insignificant corrections to the formulas developed in the preceding section. However, the partial derivatives with respect to $w$ give contributions in the direction of the dispersion on the plate, and they must be evaluated.

The expression of formula (10d+11d) is as follows:

$$F_4+F_4'=\tfrac{1}{2}w\cdot l^2\left[\frac{\sin\alpha}{r}\left(\frac{1}{r}+\frac{\cos\alpha}{R}\right)+\frac{\sin\beta}{r'}\left(\frac{1}{r'}-\frac{\cos\beta}{R}\right)\right]+\frac{w\cdot\sin\alpha}{2r^2}(z^2-2lz)+\frac{w\cdot\sin\beta}{2r'^2}(z'^2-2lz'). \quad (54)$$

The introduction of the values for the Rowland circle from Eq. (17) to eliminate $r$, gives

$$F_4+F_4'=\frac{w\cdot l^2}{2R^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)-\frac{w\cdot l}{R^2}\left(z\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'\cdot\frac{\sin\beta}{\cos^2\beta}\right)+\frac{w}{2R^2}\left(z^2\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'^2\cdot\frac{\sin\beta}{\cos^2\beta}\right). \quad (55)$$

The values for $l$, $z$ and $z'$ are not independent of each other, but are connected by the earlier formula (51) for the astigmatism, by which any one of the three can be eliminated. The elimination of $z$ is equivalent to an assumption that the slit is indefinitely long, its effective length being determined by the light rays which pass it and which strike the grating within the length $l$ of the groove in such manner that diffracted light arrives at the focal curve within the height $z'$. Since $z'$ and $l$ remain in the equation, this application of Eq. (51) may be called the case of an indefinitely long slit, but with fixed values of $l$ and $z'$.

#### (a) The Case of a Long Slit

To carry out the elimination of $z$, Eq. (51) must be evaluated for $z$, and this expression introduced into Eq. (55), where $z$ appears in the second term and $z^2$ in the third one inside the parentheses on the right-hand side. Since these substitutions become quite lengthy, they will be carried out successively for the individual members. The second term of Eq. (55) is transformed by introducing Eq. (51), as follows:

$$-\frac{w\cdot l^2}{R^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^2\beta\cdot\sin\alpha}{\cos\beta\cdot\cos\alpha}\right)+\frac{w\cdot l}{R^2}\cdot z'\cdot\frac{\sin\alpha}{\cos\alpha\cdot\cos\beta}-\frac{w\cdot l}{R^2}\cdot z'\cdot\frac{\sin\beta}{\cos^2\beta}$$
$$=-\frac{w\cdot l^2}{R^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^2\beta\cdot\sin\alpha}{\cos\beta\cdot\cos\alpha}\right)+\frac{w\cdot l}{R^2}\cdot z'\cdot\left(\frac{\sin\alpha}{\cos\alpha\cdot\cos\beta}-\frac{\sin\beta}{\cos^2\beta}\right).\quad(56)$$

Into the third member of Eq. (55) $z^2$ as calculated from Eq. (51) is introduced. The result is the following expression:

$$\frac{w}{2R^2}\left(z^2\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'^2\frac{\sin\beta}{\cos\beta}\right)$$
$$=\frac{w}{2R^2}\cdot l^2\cdot\left(\frac{\sin^5\alpha}{\cos^2\alpha}+2\frac{\sin^3\alpha\cdot\sin^2\beta}{\cos\alpha\cdot\cos\beta}+\frac{\sin\alpha\cdot\sin^4\beta}{\cos^2\beta}\right)+\cdots+\cdots.\quad(57)$$

The plus signs at the right end of (57) indicate that members in $z'$ and $z'^2$ have been omitted. This simplification in effect restricts the consideration to the coma in the Rowland plane only, because then $z'$ becomes equal to zero. Omitting therefore all the members with $z'$ and $z'^2$ and introducing (56) and (57) into (55), the result is as follows:

$$F_4+F_4'=\frac{w\cdot l^2}{2R^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)-\frac{w\cdot l^2}{2R^2}\cdot\left(2\cdot\frac{\sin^3\alpha}{\cos^2\alpha}+2\frac{\sin^2\beta\sin\alpha}{\cos\beta\cdot\cos\alpha}\right)$$
$$+\frac{w\cdot l^2}{2R^2}\left(\frac{\sin^5\alpha}{\cos^2\alpha}+2\frac{\sin^3\alpha\sin^2\beta}{\cos\alpha\cdot\cos\beta}+\frac{\sin\alpha\cdot\sin^4\beta}{\cos^2\beta}\right).\quad(58)$$

After appropriate transformation and combination of the trigonometric functions, this expression becomes:

$$F_4+F_4'=\frac{w\cdot l^2}{2R^2}(-\sin^3\alpha+\sin\beta\cdot\tan^2\beta+\sin\alpha\cdot\sin^2\beta\cdot\tan^2\beta-\sin 2\alpha\cdot\sin\beta\cdot\tan\beta).\quad(59)$$

The introduction of Eq. (51) into the computation involves a restriction in the slit length $z$, because points $z'$ on the plate receive from a grating with ruled grooves of the length $l$ only the light that originated over a certain range of $z$ at the slit. This restriction is still effective for $z'=0$, the Rowland plane. The Eq. (59) expresses the fact that a limitation of $F_4+F_4'$ requires a limitation of $l$, whereby the permissible illumination of the slit becomes, indirectly, more restricted.

The partial derivative $\partial(F_4+F_4')/\partial w$ designates the angles by which the rays from points along $w$ deviate from the true directions towards the focus $B$. These angles are projected onto the plate as length deviations $\Delta p_c$ in the direction of dispersion. The values $\Delta p_c$ can be calculated as follows:

$$\Delta p_c=\frac{\partial(F_4+F_4')}{\partial w}\cdot R\cdot\cos\beta\cdot\frac{1}{\cos\beta}=R\cdot\frac{\partial(F_4+F_4')}{\partial w},\quad(60)$$

where $R\cdot\cos\beta$ is the distance of the focus $B$ from the grating, and $1/\cos\beta$ takes care of the angle $\beta$ of oblique incidence on the plate. The combination of Eqs. (59) and (60) yields the final formula:

$$\Delta p_c=\frac{\partial(F_4+F_4')}{\partial w}\cdot R=\frac{l^2}{2R}(-\sin^3\alpha+\sin\beta\tan^2\beta+\sin\alpha\cdot\sin^2\beta\tan^2\beta-\sin 2\alpha\cdot\sin\beta\cdot\tan\beta).\quad(61)$$

The angular part of this function has been evaluated for the whole range of $\alpha$ and $\beta$. For the use in practice, however, Eq. (61) is not the most convenient, because usually the requirement is not to calculate the amount of coma present, but rather the maximum length of the rulings that can be used for a particular resolving power within the limits imposed by the amount of coma. The resolving power desired depends of course on the problem at hand. Even the highest resolution, however, permits a value of $\Delta p_c$ of one-third of the diffraction width of the lines,—an amount that reduces the theoretical resolving power by only 8 percent. The value of $\Delta p_c$ can be determined in cm for the requirements of any problem, by considering the desired line-width and resolution at the plate. The question as to the length of the grooves that can be utilized is then answered by the equation:

$$l=(2R\cdot\Delta p_c)^{\frac{1}{2}}\cdot\nu, \tag{62}$$

where $\nu$ is defined by Eqs. (61) and (62), as follows:

$$\frac{1}{\nu^2}=(-\sin^3\alpha+\sin\beta\cdot\tan^2\beta +\sin\alpha\cdot\sin^2\beta\cdot\tan^2\beta -\sin 2\alpha\cdot\sin\beta\cdot\tan\beta). \tag{63}$$

The angular function $\nu$ has been evaluated numerically and is represented in Fig. 12 for all values of $\alpha$ and $\beta$. For practical use, it is only necessary to multiply the numbers in the diagram by $(2R\cdot\Delta p_c)^{\frac{1}{2}}$; the product then represents the permissible length of the rulings $l$ as measured from the "equator" of the grating.

An example will illustrate this calculation. The question may be asked, for what wave-length range a 30,000 line grating with 15.0×7.5 cm ruled surface and 30 foot radius of curvature can be used with full aperture and an angle of incidence of 20°, if the slit is long. The evaluation of the diffraction width $\Delta p$ of the spectral lines in the region of $\beta=30°$ yields $\Delta p=0.025$ mm (independent of the order); therefore $\Delta p_c$ can have any value up to $\frac{1}{3}$ of this value or to $\Delta p_c=0.008$ mm. The numerical factor is determined by the following expression:

$$(\Delta p_c\cdot 2R)^{\frac{1}{2}}=(8\cdot 10^{-4}\cdot 2\cdot 914)^{\frac{1}{2}} =(1.46)^{\frac{1}{2}}=1.21\text{ cm}. \tag{64}$$

The allowed length and the function $\nu$ are limited as follows:

$$l\leqslant 1.21\nu;\quad \nu\geqslant\frac{l}{1.21}.$$

In this equation, the grooves are measured from the equator; that means for the grating chosen $l=3.75$ cm, and consequently $\nu\geqslant 3.10$. In the diagram Fig. 12 the values of $\nu$ for the present case are found along the straight line for $\alpha=20°$. It is obvious that $\nu$ is larger than 3.10 for the values of $\beta$ corresponding to first-order wave-lengths from 500 to 7900A. Within this range of $\beta$, the grating may be used at its full height. Outside of this range, the length of the rulings must be decreased, lest the resolving power be damaged.

### (b) The Coma for a Short Slit

In many cases it is desirable and possible to use a long slit, in order to counteract the loss in illumination on the plate caused by the astigmatism. In other cases, it is not advantageous to illuminate such a long segment of the slit and on the other hand the length of the rulings may increase the illumination also. Therefore, the coma in its dependence on the length of the slit will be evaluated. It can be shown that frequently the illuminated segment of the slit is too short to justify the application of the formulas of the preceding section.

The evaluation for this case means mathematically that not $z$, but $l$ must be eliminated from Eq. (55) by means of Eq. (51). The rulings are now considered to be indefinitely long, but only that length $l$, which is determined by Eq. (51), contributes to the image formation at $z'$, caused by light at $z$. The expression for $l$ is as follows:

$$l=\frac{\dfrac{z}{\cos\alpha}+\dfrac{z'}{\cos\beta}}{\dfrac{\sin^2\alpha}{\cos\alpha}+\dfrac{\sin^2\beta}{\cos\beta}}. \tag{65}$$

The introduction of Eq. (65) into Eq. (44) yields an expression for $(F_4+F_4')$, which can be simplified if only points in the Rowland plane ($z'=0$) are considered, and the attempt is not made to calculate coma for points $z'$ that are away from the equatorial plane. Omitting the stepwise introduction of Eq. (65) into Eq. (55), as was carried out in the Eqs. (56) and (57), the final equation for the Rowland plane with $z'=0$, is:

$$F_4+F_4'=\frac{w}{2R^2}\cdot z^2\cdot\frac{1}{\cos^2\alpha\left(\dfrac{\sin^2\alpha}{\cos\alpha}+\dfrac{\sin^2\beta}{\cos\beta}\right)^2}\times\left[\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)-2\cdot\frac{\dfrac{\sin\alpha}{\cos^2\alpha}}{\cos\alpha\left(\dfrac{\sin^2\alpha}{\cos\alpha}+\dfrac{\sin^2\beta}{\cos\beta}\right)}+\frac{\sin\alpha}{\cos^2\alpha}\right].\quad(66)$$

Or

$$F_4+F_4'=\frac{w\cdot z^2}{2R^2\left(\sin^2\alpha+\sin^2\beta\dfrac{\cos\alpha}{\cos\beta}\right)^2}\times(-\sin^3\alpha+\sin\beta\cdot\tan^2\beta+\sin\alpha\cdot\sin^2\beta\cdot\tan^2\beta-\sin 2\alpha\cdot\sin\beta\cdot\tan\beta).\quad(67)$$

The angular function in the denominator is the square of the expression that occurs in the formulae (45), (51) for the astigmatism. By Eq. (46), it has been defined as $\Gamma$. The sum in the parenthesis in the numerator is the expression $\nu$ of Eqs. (59) and (63) for the coma. In order to calculate the deviation $\Delta p_s$, that is caused by the members $(F_4+F_4')$ at the plate in the direction of the dispersion, the same reasoning that yielded the Eq. (60) may be applied with the following result:

$$\Delta p_s=\frac{\partial(F_4+F_4')}{\partial w}\cdot R=\frac{z^2}{2R}\cdot\frac{1}{\Gamma^2}\cdot\frac{1}{\nu^2}.\quad(68)$$

The significance of this equation is, that the length of the slit must be limited in order to reduce the width of the coma to a value that does not damage the resolution at $z'=0$, if the grooves of the grating are "very long;" that is long enough to receive all the light coming from the slit along $z$ that contributes to the image at $z'=0$. For practical use, the question is frequently asked as to how long a slit is allowable for a permissible amount $\Delta p_s$ of coma. In analogy to the transformation in the preceding section, the result is as follows:

$$z\leqslant(2R\cdot\Delta p_s)^{\frac{1}{2}}\cdot\Gamma\cdot\nu.\quad(69)$$

The angular function $\Gamma\cdot\nu$ has been calculated

Coma
The allowed length of the grooves for the coma $\Delta c$ at $z'=0$ can be calculated by $l=\gamma\sqrt{\Delta c\cdot 2R}\,\gamma$, $\gamma$ being represented in the graph.

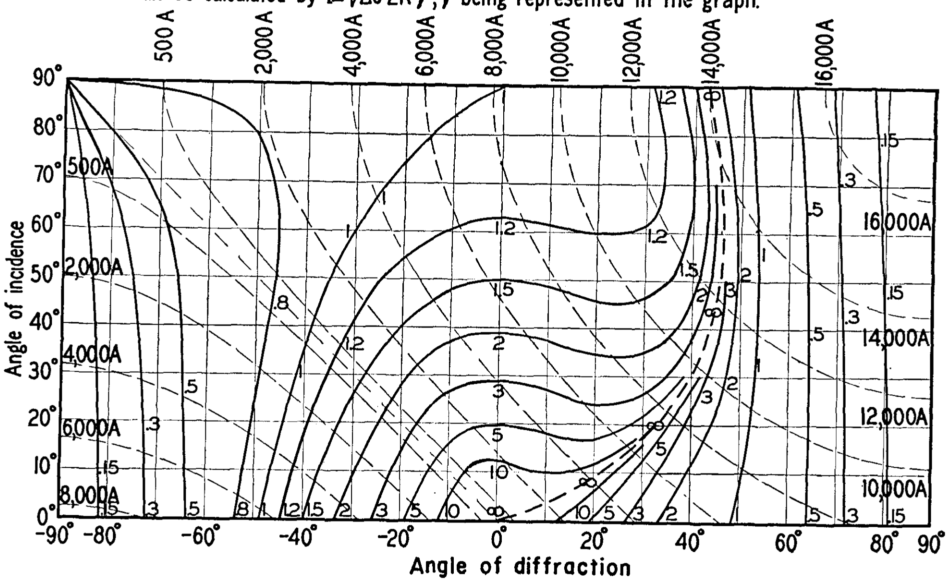


FIG. 12. Allowed length of grooves for coma $\Delta pc$ in units of $(2R\cdot\Delta pc)^{\frac{1}{2}}$ for all angles of incidence and diffraction.

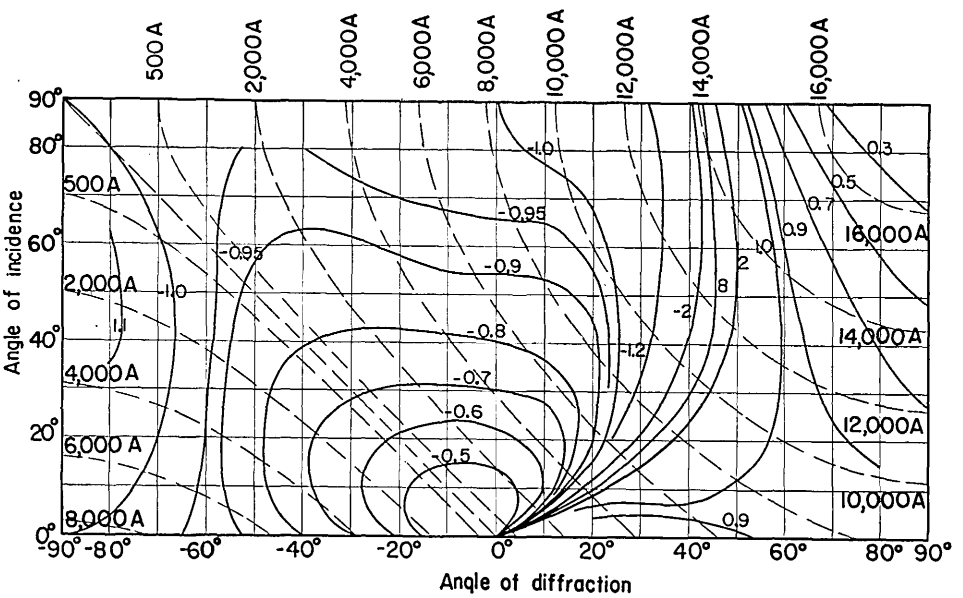


FIG. 13. Allowed length of slit for coma $\Delta ps$ in units of $(2R\cdot\Delta ps)^{\frac{1}{2}}$ for all angles of incidence and diffraction.

for the whole range of $\alpha$ and $\beta$, and it is represented by the diagram in Fig. 13, in which the numbers indicate its numerical values. For practical application, it is necessary to assign a value to the permissible amount of coma. This value may be determined by the resolving power in the case of large circle mountings, or by the plate grain in the case of shorter focus, or by the maximum allowable line width for photometry or other purposes. The introduction of $\Delta p_s$ and $R$, and of the value $\Gamma\cdot\nu$ for the angles $\alpha$ and $\beta$ into Eq. (69) yields the permissible length of the slit, measured from its center, $z=0$. The positive or negative signs of the original angular function in Eq. (67) are not given in the diagram—they mean that the coma extends to one or to the other side of the spectral lines.

### (c) General Conclusions Regarding Coma

The question arises, which of the limitations, slit length or length of the rulings, must be applied to any special case? The answer is very simply given by the method of the derivation, which in both cases assumed the other variable to be indefinitely extended. If from the diagram Fig. 12 it is found that the exposed grooves of the grating are not too long, then it is not necessary to restrict the length of the illumination at the slit. If, on the other hand, the slit length is restricted to that allowed by Eq. (69) and Fig. 13, then no limitations are placed on the length of the rulings. If, however, both slit and rulings in the experimental conditions are longer than is permissible, the best compromise seems to be to reduce both lengths until they yield the same amount of coma. A more precise method, of course, is the evaluation of Eq. (55) for the chosen values of $l$ and $z$, setting $z'=0$. If the range of $\alpha$ and $\beta$ for which a knowledge of the coma is wanted, is not too large, the work of calculation is not extensive. The partial derivative of $(F_4+F_4')$ then yields the coma, according to Eq. (60).

Numerical calculation of the coma at values $z'=0$ is, however, rather complicated, if it has to be carried out over larger ranges of $\alpha$ and $\beta$. However, it is possible to approximate this problem by the diagrams for the curvature of the spectral lines as given in the next section. There is no simple general rule for the increase or decrease of the coma if $z'$ increases; of course, the coma is symmetrical with respect to $+z'$ and $-z'$. For the angular regions of large astigmatism, the coma usually has a maximum at $z'=0$; for regions of small astigmatism, it has its minimum there.

The limitation of the length of the rulings of course reduces the used aperture and thereby the intensity of the grating. Increasing the length of the slit, an operation that has sometimes been recommended as a remedy against the intensity loss caused by the astigmatism, is also limited by coma. Coma can be partially reduced by using a curved slit, but a change of the curvature with change of the angles $\alpha$ and $\beta$ is required.

The diagrams show the ranges for which the coma is least disturbing. The Rowland and Abney mountings are not favorable for long grooves. The Paschen-Runge mounting is good, if the angle of incidence is not the usual one of approximately 45°, but only half of its value or less. The Eagle mounting is very good. Since for small angles the coma is always not too disturbing, there is only a slight preference for the alternative of placing the slit on the side of the plateholder that is nearer the normal. For larger angles and wave-lengths, however, a great advantage is gained by using the slit on the side of the plateholder away from the normal, or with $\alpha$ larger than $\beta$.

## IX. THE CURVATURE OF THE SPECTRAL LINES

As already mentioned in the introductory remarks of the last section, the rulings of the grating are seen as curves from a point at the slit, whenever the angle of incidence is larger than zero. Conversely, the grooves appear curved from any point on the plate holder except for $\beta=0$. The simple geometrical reflection of light from a point source $A$ at a set of rulings therefore produces a curved sheet of light. Before this sheet converges to the secondary focus, it is intercepted by the cylindrical plateholder. The same holds for diffracted light of any wavelength. The line of intersection—the spectral line—is therefore curved. These curved spectral lines, are the astigmatic images of a point source at the slit. Therefore this curvature will be called the astigmatic curvature.

In addition to this curvature, there is another kind of curvature in the spectral lines, that is caused by the finite length of the slit. The vertices of the astigmatic images, correspond to a point source at the slit, with $z=0$. If the point source is moved along the slit, then the vertex of this "parabola" will describe a curved line or envelope which must also be investigated. This curvature is quite different from the astigmatic curvature; it may even be concave to the opposite side. This enveloping curvature will be treated in a separate section. The two effects must be added to evaluate the complete curvature.

### (a) The Astigmatic Curvature

The astigmatic curvature of the spectral lines arises from the cross products containing $w$ and $l$ in the general series development of Eqs. (10) and (11). The largest members of this type are $F_4$ and $F_4'$, the next ones are $F_8$ and $F_8'$. The analytic expression for any curvature must contain the dependence of the angle $\beta$ on the coordinate $z'$, which is the distance along the spectral line measured from the Rowland plane $z'=0$. The dependence of $\beta$ on $z'$ means that the partial derivative $\partial(F_4+F_4')/\partial w$ assumes different values with varying $z'$, since that partial derivative designates the differences $d\beta$ between the actual directions of the normals to the wave fronts and the direction towards $B$ (Fig. 1), and, if we consider the astigmatism arising from $\partial(F_3+F_3')/\partial l$, towards the points along the vertical through $B$, these all have the same angle $\beta$.

If any partial derivative $\partial(F_n+F_n')/\partial w$ should contain an odd power of $z'$, an inclination of the spectral line would result. The dependence on $(z')^2$ however yields a curved line, in general a parabola. The angles $d\beta$ are projected onto the Rowland cylinder as distances $\Delta p$ according to the Eq. (60) used earlier.

$$\Delta p=d\beta\cdot R\cdot\cos\beta\cdot\frac{1}{\cos\beta}=R\cdot\frac{\partial(F_4+F_4')}{\partial w}. \quad (60)$$

The difference between the evaluation of the Eq. (60) for coma and of that for curvature consist in the different choice of the independent variables in $F_4+F_4'$. For the coma, it was assumed that $z'=0$; now the same equation is evaluated for variable values of $z'$, with $z$ restricted to the value zero.

The starting point is Eq. (55) for the Rowland circle,

$$F_4+F_4'=\frac{w\cdot l^2}{2R^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)$$
$$-\frac{w\cdot l}{R^2}\left(z\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'\cdot\frac{\sin\beta}{\cos^2\beta}\right)$$
$$+\frac{w}{2R^2}\left(z^2\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'^2\cdot\frac{\sin\beta}{\cos^2\beta}\right). \quad (55)$$

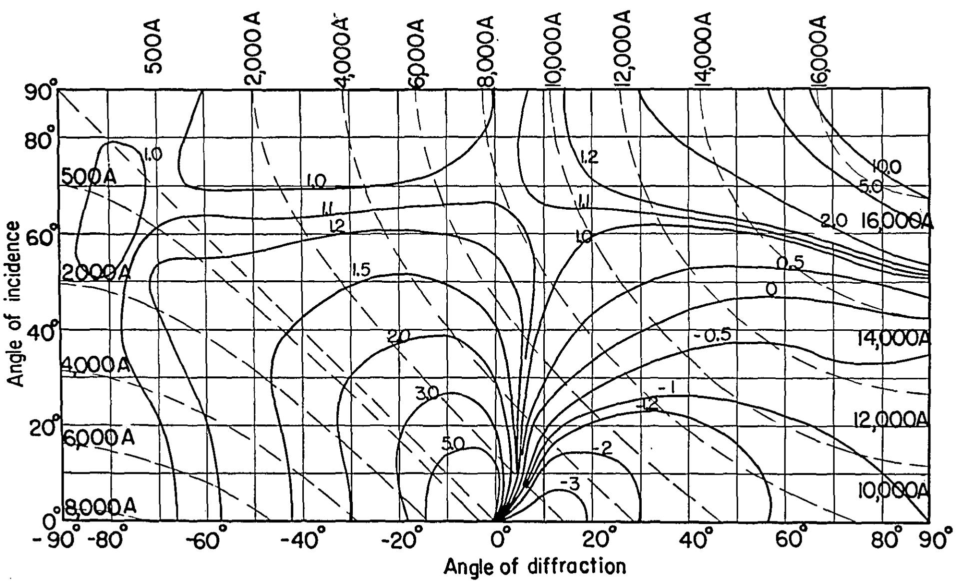


FIG. 14. Displacement of spectral line by astigmatic curvature at a distance $z'$ from the Rowland plane in units of $(z')^2/2R$ for all angles of incidence and diffraction.

Here, $l$, $z$ and $z'$ are not independent from each other, since the condition (36) must also be fulfilled,

$$\partial(F_3+F_3')/\partial l=0. \tag{36}$$

The result for the special case $z=0$ has been derived earlier, as follows:

$$z'=l\left(\sin^2\beta+\sin^2\alpha\cdot\frac{\cos\beta}{\cos\alpha}\right). \tag{45}$$

It has been written in an abbreviated form,

$$l=z'/\Gamma. \tag{46}$$

This formula can be used to eliminate $l$, so that $z'$ will be the only vertical coordinate left. For $z=0$, the result is, as follows:

$$\begin{aligned}F_4+F_4'=&\frac{w\cdot z'^2}{2R^2\cdot\Gamma^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)\\&-\frac{w\cdot z'^2}{R^2\cdot\Gamma}\cdot\frac{\sin\beta}{\cos^2\beta}+\frac{w\cdot z'^2}{2R^2}\cdot\frac{\sin\beta}{\cos^2\beta}\\=&\frac{w\cdot z'^2}{2R^2\cdot\Gamma^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)\\&-2\Gamma\cdot\frac{\sin\beta}{\cos^2\beta}+\Gamma^2\cdot\frac{\sin\beta}{\cos^2\beta}.\end{aligned} \tag{70}$$

$$F_4+F_4'=\frac{w\cdot z'^2}{2R^2\cdot\Gamma^2}\left[\frac{\sin^3\alpha}{\cos^2\alpha}-\sin\beta+\frac{\sin\beta}{\cos^2\beta}(1-\Gamma)^2\right]. \tag{71}$$

In order to obtain the equation of the curvature, the angular deviation $d\beta$ must be calculated, which is given by the partial derivative $\partial(F_4+F_4')/\partial w$, and the result multiplied by $R$, as in Eq. (60). The result is as follows:

$$\Delta p_a=\frac{\partial(F_4+F_4')}{\partial w}\cdot R=\frac{z'^2}{2R\cdot\Gamma^2}\times\left[\frac{\sin^3\alpha}{\cos^2\alpha}-\sin\beta+\frac{\sin\beta}{\cos^2\beta}(1-\Gamma)^2\right]. \tag{72}$$

Equation (72) represents a family of parabolas, the vertices of which lie at $z'=0$, i.e., in the equatorial plane. Since $\sin\beta$ occurs in the first power, the curvatures of the parabolas will change their signs along the Rowland circle; they degenerate to straight lines for the case that the expression in the parenthesis becomes zero.

The angular function of Eq. (72), which has been denoted by $\psi$, is given in Eq. (73). It has been numerically evaluated for the whole range of $\alpha$ and $\beta$; the results are represented in the diagram Fig. 14. For practical use, the numbers as read from the diagram have to be multiplied by $z'^2/2R$, the product being the displacement of the spectral line at $z'$ from the value at $z'=0$.

$$\Psi=\frac{1}{\Gamma^2}\left[\frac{\sin^3\alpha}{\cos^2\alpha}-\sin\beta+\frac{\sin\beta}{\cos^2\beta}(1-\Gamma)^2\right]. \quad (73)$$

From the diagram it is obvious, that the highest curvature occurs in the region of the Eagle mounting. For a Paschen-Runge mounting, the spectral lines change their direction of curvature three times. This behavior must be taken into account, if precision wave-length measurements are to be made at different "heights" of the spectral lines, especially if a mask has been used for obtaining several spectra on the same plate, and these spectra are to be related to each other.

### (b) The Enveloping Curvature

The astigmatic curvature comprises only one part of the dependence of the height $z'$ on the angle $\beta$; it is preponderant only for the case of a short slit and large astigmatism, which is found chiefly at grazing incidence. If however the astigmatism is small, then the length of the mentioned parabolas is small or negligible. Under these conditions the enveloping curvature becomes important.

The different points of the slit will give rise to different astigmatic images, and the center of each one is determined in height by a relation that can be derived simply from geometry, as follows:

$$z:r=z':r',$$

$$z=z'\cdot\frac{\cos\alpha}{\cos\beta};\quad z'=z\cdot\frac{\cos\beta}{\cos\alpha}. \quad (74)$$

This equation can be used to eliminate $z$ from Eq. (55), and if in addition the equations for the astigmatism are applied to eliminate $l$, then a curvature $\Delta p_b=f(z'^2)$ will be obtained that gives the locus for the centers of the astigmatic images. It can be shown, that for $z\neq 0$ each astigmatic image is also a parabola, very similar to that for $z=0$ at the same angles $\alpha$ and $\beta$. Therefore the curvature $\Delta p_b=f(z'^2)$ connects the vertices of the parabolae for the different points $z$ at the slit.

The start of the calculation is again Eq. (55)

$$F_4'+F_4'=\frac{w\cdot l^2}{2R^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)-\frac{w\cdot l}{R^2}\left(z\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'\frac{\sin\beta}{\cos^2\beta}\right)+\frac{w}{2R^2}\left(z^2\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'^2\cdot\frac{\sin\beta}{\cos^2\beta}\right). \quad (55)$$

The astigmatism must be introduced in the formulation of Eq. (51), containing the coordinate $z$. Using the abbreviation of Eq. (46):

$$z'=l\cdot\Gamma-z\frac{\cos\beta}{\cos\alpha};\quad l=\frac{1}{\Gamma}\left(z'+z\cdot\frac{\cos\beta}{\cos\alpha}\right). \quad (75)$$

Combining this equation with (74) yields the equation:

$$l=2z'/\Gamma. \quad (76)$$

By using these substitutions, $z'$ is the only vertical coordinate left in the equation, as follows:

$$F_4+F_4'=\frac{w\cdot(2z')^2}{2R^2\cdot\Gamma^2}\left(\frac{\sin^3\alpha}{\cos^2\alpha}+\frac{\sin^3\beta}{\cos^2\beta}\right)-\frac{w\cdot 2z'}{R^2\cdot\Gamma}\times\left(z'\cdot\frac{\cos\alpha}{\cos\beta}\cdot\frac{\sin\alpha}{\cos^2\alpha}+z'\cdot\frac{\sin\beta}{\cos^2\beta}\right)+\frac{w\cdot z'^2}{2R^2}\left(\frac{\cos^2\alpha}{\cos^2\beta}\cdot\frac{\sin\alpha}{\cos^2\alpha}+\frac{\sin\beta}{\cos^2\beta}\right). \quad (77)$$

This formula can be simplified as follows:

$$F_4+F_4'=\frac{w\cdot z'^2}{2R^2\cdot\Gamma^2}\left[4\frac{\sin^3\alpha}{\cos^2\alpha}+4\frac{\sin^3\beta}{\cos^2\beta}-4\cdot\Gamma\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}\right)\frac{1}{\cos\beta}+\Gamma^2\cdot\frac{\sin\alpha+\sin\beta}{\cos^2\beta}\right]$$

$$=\frac{w\cdot z'^2}{2R^2\cdot\Gamma^2}(\sin\alpha+\sin\beta)\times\left(\frac{\Gamma^2}{\cos^2\beta}-4\cdot\tan\alpha\cdot\tan\beta\right). \quad (77a)$$

$$=\frac{w\cdot z'^2}{2R^2}(\sin\alpha+\sin\beta)\times\left(\frac{1}{\cos^2\beta}-4\cdot\frac{\tan\alpha\cdot\tan\beta}{\Gamma^2}\right). \quad (78)$$

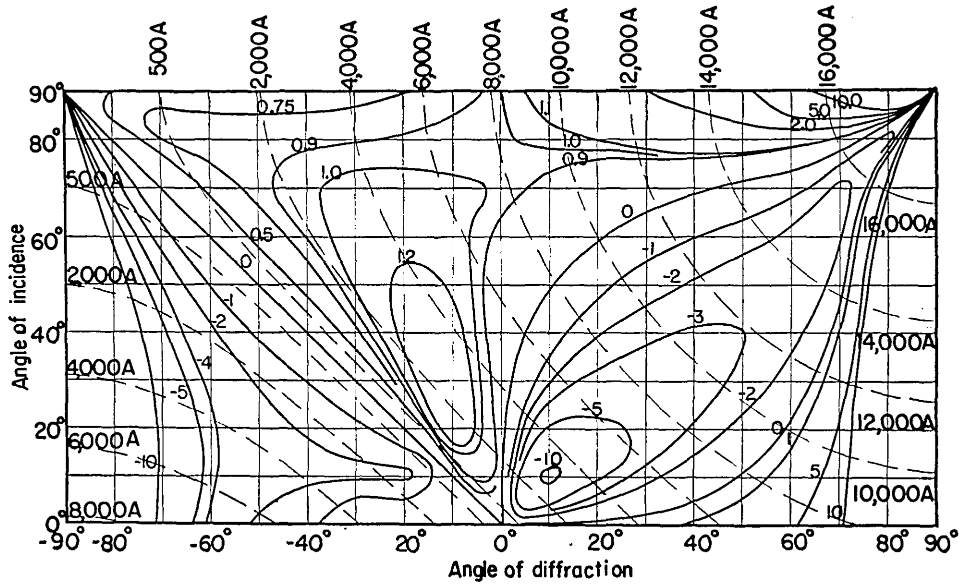


FIG. 15. Displacement of spectral line by enveloping curvature at a distance $z'$ from the Rowland plane in units of $(z')^2/2R$ for all angles of incidence and diffraction.

The curvature at the plate can be obtained by analogy to Eq. (60) as the product of $R$ with the partial derivative of Eq. (78) with respect to $w$, as follows:

$$\Delta p_b = \frac{\partial(F_4+F_4')}{\partial w}\cdot R = \frac{z'^2}{2R}(\sin\alpha+\sin\beta) \times\left(\frac{1}{\cos^2\beta}-4\frac{\tan\alpha\cdot\tan\beta}{\Gamma^2}\right). \quad (79)$$

The result is again a family of parabolas, the vertices of which lie in the equatorial plane $z'=0$. Their curvature changes sign repeatedly. The angular function $\phi$ may be defined as follows:

$$\Phi = (\sin\alpha+\sin\beta)\left(\frac{1}{\cos^2\beta}-4\frac{\tan\alpha\cdot\tan\beta}{\Gamma^2}\right). \quad (80)$$

The numerical values of $\Phi$ have been calculated; they are represented in the diagram Fig. 15. For the evaluation of any practical case, it is only necessary to read the number for the values $\alpha$ and $\beta$ of the particular spectral line from the diagram and to multiply by $z'^2/2R$. The product is the displacement (in cm) of the line at $z'$ from the position that it has at $z'=0$.

## X. BROADENING OF THE SPECTRAL LINES BY ABERRATION

Spherical aberration not only causes light of the wrong phase to fall on the focal point $B$, but also it causes light to reach points which lie on both sides of $B$. This broadening of the spectral lines can be calculated by the formulas (24) and (27). For the sidewise deviation the Eq. (6a) is applied. The partial derivative $\partial F/\partial w$ designates the angles $\Delta\beta$ that the real rays arising from points $P$ along $w$ form with the geometrical directions from them towards the focus $B$. These deviations from the desired directions are projected themselves as distances $\Delta p$ from the true positions on the plate. From Eqs. (6c) and (17) and from the relation for the Rowland circle $\beta=\gamma$, the magnitude $\Delta p$ is determined, as follows:

$$\Delta p = \frac{\partial(F_5+F_5')}{\partial w}\cdot\frac{R\cdot\cos\beta}{\cos\beta} = R\frac{\partial(F_5+F_5')}{\partial w}. \quad (81)$$

By introducing the Eqs. (24) and (25) for the aberration, there follows:

$$\Delta p = R\frac{\partial(F_5+F_5')}{\partial w} = \frac{1}{8R^2}\frac{d(w^4+2w^2l^2+l^4)}{dw}\cdot\phi. \quad (82)$$

The result for the Rowland circle can be obtained easily, by using Eq. (27), as follows:

$$\Delta p = \frac{1}{8R^2}[4w(w^2+l^2)]\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}\right). \quad (83)$$

There is a well known example in the literature for the broadening of the spectral lines by aberration. It is a photograph taken by Edlén[26] of some lines in the vacuum ultraviolet, that gave rise to the theoretical considerations of Mack, Stehn, and Edlén[27] who attempted no calculation of the broadening of the lines. Attention was directed only to the loss in resolving power and to the necessary restriction of the width of the grating. The formula (83), however, allows a quantitative calculation of the widths of the so-called "ghosts" of the photograph.

The experimental data can be taken from Edlén's paper. A grating with 1183 lines per mm, and 1 m radius of curvature was used at angle of incidence $\alpha=84.4°$. The lines shown in the photograph have wave-length of about 279A. Their broadening by excessive aperture of the grating can be determined, because the wave-lengths of five lines are given. The strong lines are widened by about 3.0A towards long wave-lengths and by about 2.7A towards short wave-length.

Applying the generalized theory to this case, the optimum aperture can be determined by Eq. (20), which is evaluated with the help of Table I and the diagram Fig. 4. The calculation of the angle $\beta$ by Eq. (14) yields $\beta=74°\ 12'$. Therefore:

$$\rho=\left(\frac{2R^3d'}{m}\right)^{\frac{1}{4}}y=3.59\times 0.22=0.790\text{ cm.}$$

The evaluation of Eq. (83) for this aperture of the grating gives a half-width $\Delta\lambda$ for the spectral lines: $\Delta\lambda=2.51\times 10^{-3}$ mm. The dispersion in this spectral region is $dl/d\lambda=0.23$ mm/$A$;[28] the line half-width $\Delta p_\lambda$ is therefore 0.014A as determined by aberration. The resolving power, however, is 19200 (equal to the number of rulings) and causes a half-width of the diffraction pattern of .014A. The result is, that the definition of the lines is controlled by the diffraction pattern, as was assumed in the calculation of the allowed width of the grating in Eq. (27).

Edlén increased the aperture of his grating stepwise up to $2\rho=7.6$ cm, corresponding to 4.82 times the width that formula (29) permits. The broadening caused by aberration becomes then $(4.82)^2=23.2$ times larger; the diffraction width, however, would decrease by the factor 4.82 if perfect focusing could be attained. Applying this factor to the earlier result, $\Delta p_\lambda=0.014$A, $\Delta p_E=3.2$A is obtained for the broadening in the experiment of Edlén.

The experimental values of 3.0 and 2.7$A$ are in agreement with the theory; the slight discrepancy at the short wave-length side may be explained by some masking effect of the grating holder. The theory could hardly be expected to check better with a single experiment, which was not intended for the purpose of quantitative evaluation.

## XI. THE IMAGE FORMATION AND ITS IMPERFECTIONS FOR THE GRATING MOUNTED IN PARALLEL LIGHT (WADSWORTH MOUNTING)

The large astigmatism of the spectra in Rowland's mounting was soon recognized as a disadvantage, especially because of the loss in intensity and of the difficulty in introducing a comparison spectrum. F. L. O. Wadsworth[29] found that stigmatic images can be obtained from a concave grating illuminated by parallel light from a collimator. Wadsworth used his original instrument, with a small five-foot grating, only for visual observations with eyepiece on the grating normal, and he constructed a mechanical device that kept the eyepiece in focus as the angle of incidence was changed. In the following year, Runge and Paschen[30] published an extensive study of the series spectra of oxygen, sulphur and selenium, photographed with a 6.5 m concave grating (20,000 lines/inch), which was mounted in parallel light. Apparently, Runge and Paschen invented the stigmatic mounting independently. They applied this mounting to the study of the radiation emitted end-on by Geissler tubes, and emphasized the reduction of dispersion and focal length and the increase in aperture inherent in this mounting as compared with the Rowland if a grating of given size is used. The feature of stigmatic image formation is not stressed in their paper, although the use with Geissler tubes viewed end-on shows that Paschen

[26] B. Edlén, Thesis, Upsala 1932.
[27] See reference 16.
[28] R. A. Sawyer, *Experimental Spectroscopy* (Prentice-Hall, Inc., New York, 1944) p. 131.
[29] F. L. O. Wadsworth, Astrophys. J. **3**, 54 (1896).
[30] C. Runge and F. Paschen, Ann. d. Physik **61**. 641 (1897).

and Runge were well aware of this fact. Fabry and Buisson[31] introduced a concave mirror instead of a lens, to avoid the absorption and chromatic aberrations of lenses. Meggers and Burns[32] described an instrument with a spherical mirror that is in permanent use at the Bureau of Standards. A. Poritzky[33] replaced the concave mirror by a paraboloidal mirror, that throws the light back along its axis to a perforated plane mirror at the slit, and thence to the grating.

### (a) The Focal Curves for the Wadsworth's Mounting

Equation (16a) is the condition for the disappearance of the aberration arising from the quadratic members in the characteristic function (5) for the light beams. As already mentioned, the values:

$$r=\infty \quad \text{and} \quad \beta=0°, \tag{18}$$

fulfill that equation, and simultaneously they make the higher members in Eq. (16) vanish also. Furthermore it has been pointed out that the introduction of these values into Eq. (16a) leads to the locus for the points at the normal where parallel light striking the grating at an angle $\alpha$ is focused, as

$$r'=R/1+\cos\alpha, \tag{19}$$

and that this curve is a parabola, with its focus at the center of the grating. For $\alpha=0°$, the light is focused at a distance $r=R/2$ from the grating, and for $\alpha=90°$, at $r'=R$, respectively.[34] Along this focal curve, all the wave-lengths up to $m(\lambda/d)=1(=\sin\alpha+\sin\beta=\sin 90°+\sin 0°)$ can be obtained. By combining Eq. (19) with (14), from which $\cos\alpha$ can be replaced by $(1-\sin^2\alpha)^{\frac{1}{2}}$ $=(1-m^2[\lambda^2/d])^{\frac{1}{2}}$, the expression becomes

$$r'=\frac{R}{1+(1-m^2[\lambda^2/d])^{\frac{1}{2}}}. \tag{84}$$

This equation shows the dependence of the focal length on the wave-length $\lambda$ which appears at the normal.

In order to find the focal curve for any given angle $\alpha$, as it extends to both sides of the normal, the restriction $\beta=0°$ is dropped, and Eq. (16a) is evaluated for the condition $r=\infty$ only. The result is:

$$r'=R\cdot\cos^2\beta/(\cos\alpha+\cos\beta). \tag{85}$$

This equation reduces to Eq. (19), as it should, for $\beta=0°$. The curves represented by Eq. (85) are roughly circular, but an analysis shows that they are a family of lemniscates with a parameter $\alpha$. Their common points lie in the center of the grating, for $\beta=\pm 90°$. In Fig. 16 the focal curves are shown. Only the parts in the neighborhood of the normal that give satisfactory images are drawn, in order to keep the figure simple. The wave-lengths as diffracted by a 30,000 lines per inch grating are also indicated. Those parts of the focal curves with $\beta>90°$ (behind the grating) are not closed loops, as are those before the grating, but they reach infinity twice for $\cos\alpha=-\cos\beta$; therefore the complete curves may be called hyperbolic lemniscates. The shape of the loops in front of the grating, for each curve in the family, is approximately an ellipse with small eccentricity, the longer axis lying at the normal of the grating, the two sides for $+\beta$ and $-\beta$ being symmetrical. For larger angles $\alpha$, the eccentricity of the loops becomes smaller and smaller, and for $\alpha=90°$ the transition to the circle $r'=R\cdot\cos\beta$ is complete.

There is a striking difference between these curves and the Rowland circle: the focal curves for the different angles of incidence have a different curvature, limiting the use of a machined rigid plateholder, and necessitating one with an adjustable curvature for different angles. Furthermore the dispersion at the normal ($\beta=0°$) is different for the various values of $\alpha$; it increases from one-half the value of the Rowland circle for $\alpha=0°$ to the full value for $\alpha=90°$, as will be shown in a later section. The reason for this change is found in the variable distance of the focus from the grating according to the parabolic Eq. (19).

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[31] C. Fabry and H. Buisson, J. de Physique [4] 9, 940 (1910).

[32] W. F. Meggers and K. Burns, Sci. Pap. Bur. Stand. [441] 18, 191 (1922).

[33] A. Poritzky, Proc. of the Fifth Conference on Spectroscopy, p. 38.

[34] Sometimes the opinion is expressed that the focal length of a grating in the Wadsworth mounting is one-half of that in the Rowland circle mounting. This statement is not generally correct. The distance from grating to plate in Wadsworth's mounting lies between $R/2$ and $R$, and on the Rowland circle between $R$ and about $R/10$. An example may illustrate this point—the focus for 4000A in the second order, diffracted by a 30,000 lines/inch grating, lies for the Eagle mounting at $0.88R$, for the Wadsworth mounting at $0.77R$. Hence, the dispersions in these two cases are only slightly different and the apertures of the grating in the two mountings differ only by 30 percent.

This variation has another consequence—the magnification factor for a length $z$ at the slit, that is stigmatically focused as $z'$ at the normal, increases from a value, $z'/z = R/2R'$ where $R'$ is the radius of curvature of the collimating mirror, to $z'/z = R/R'$ as increases from 0° to 90°. This magnification of course governs not only the length $z'$ of the spectral lines, but also their width, to the extent that this depends on the width of the slit; that is, for image widths wider than the diffraction pattern caused by the grating. There is another special solution of Eq. (85) worth mentioning,—the case $\alpha = \beta$, which leads to

$$r' = (R/2)\cos\beta. \tag{86}$$

This equation is that of a circle with the radius $R/4$ as the locus for the focal points. The arrangement can be visualized as a kind of Eagle mounting, in which the grating is illuminated with parallel light.

In the general Eqs. (10) and (11), the terms in $F_2$ and $F_2'$ which contain $w^3$ and $w^4$ vanish for the conditions $r = \infty$, $\beta = 0$, and since in $F_6 + F_6'$ the same angular coefficients are present, they will likewise vanish for the same conditions. On the normal to the grating, the image formation is therefore quite free from aberrations arising from the grating.

Usually, however, in grating spectroscopy, a satisfactory image formation is wanted over broad ranges of wave-length, and therefore the conditions must be investigated at which images are formed over a considerable range of $\beta$ around $\beta = 0$. The wave-lengths appearing in their dependence on $\alpha$ can be read from Fig. 16, or more exactly from the diagram Fig. 2; they are the same as those appearing in Rowland's or Abney's mounting under the same angles. The fact that the focal distance is different in the two cases, has been mentioned already. Fig. 16 represents the positions of the focal curves as a function of wave-length, as a grating with 30,000 lines/inch is turned about its axis to vary $\alpha$.

## (b) The Aberrations in the Stigmatic Mounting

### *1. The Cubic Terms*

The same mathematical conditions hold here as for the Rowland circle. The aberrations arise from the contributions of the higher members of the series development in Eqs. (10) and (11).

By introducing the conditions $r = \infty$ and $r' = R\cos^2\beta/(\cos\alpha + \cos\beta)$ (Eq. (85)) into Eq. (23), the effect of the cubic terms may be investigated for the condition of parallel light striking the grating. Equation (23) becomes

$$\frac{W^3}{2}\,\frac{\sin\beta\cdot(\cos\alpha+\cos\beta)\cdot\cos\alpha}{R^2\cdot\cos^2\beta} \leqslant \frac{\lambda}{4}. \tag{87}$$

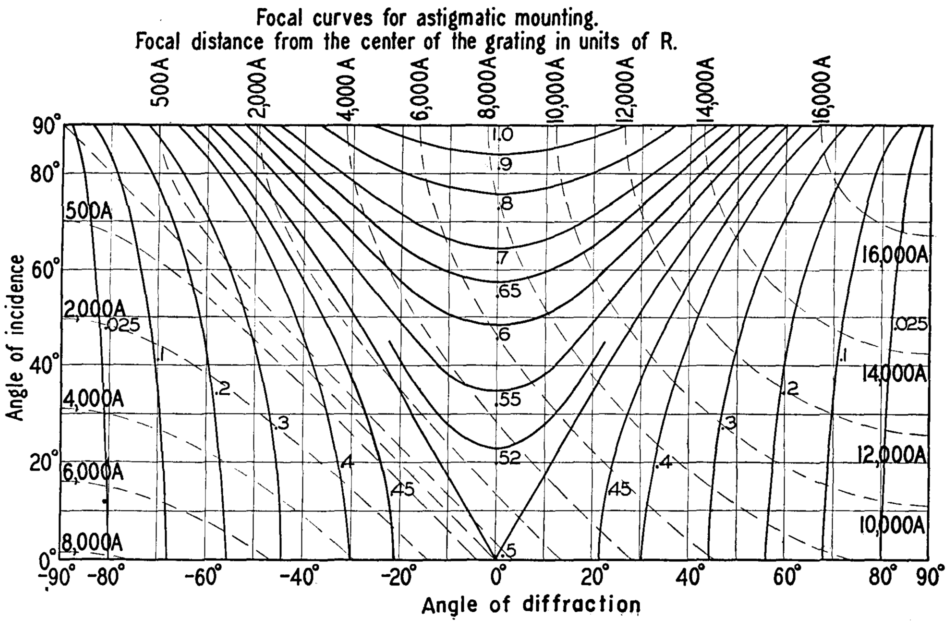


FIG. 16. Focal distance from the center of the grating in the Wadsworth mounting in units of $R$ for all angles of incidence and diffraction.

If $\lambda$ is expressed in terms of $\alpha$ and $\beta$ by use of Eq. (14), the maximum value for $W$ is given by:

$$W \leqslant \left( \frac{d \cdot R^2 (\sin \alpha + \sin \beta) \cdot \cos^2 \beta}{2m \cdot \sin \beta \cdot (\cos \alpha + \cos \beta) \cdot \cos \alpha} \right)^{\frac{1}{3}}. \quad (88)$$

After trigonometric transformation, the equation may be written:

$$W \leqslant \left( \frac{d \cdot R^2}{2m} \cdot \tan \frac{\alpha + \beta}{2} \cdot \frac{\cos^2 \beta}{\cos \alpha \cdot \sin \beta} \right)^{\frac{1}{3}}. \quad (89)$$

This expression governs the maximum width of the grating in Wadsworth's mounting. In comparison to the Rowland circle, the allowed width is rather small. The angular function

$$y_s = \left( \tan \frac{\alpha + \beta}{2} \cdot \frac{\cos^2 \beta}{\cos \alpha \cdot \sin \beta} \right)^{\frac{1}{3}}, \quad (90)$$

has been numerically evaluated for the angles $\alpha$ and $\beta$ of a grating with 30,000 lines/inch, and the results are represented in Fig. 17.

*Numerical factors in the formula for optimal width.*

The numerical values of the factor $(dR^2/2m)^{\frac{1}{3}}$ have been calculated for the more usual sizes of gratings, and are listed in the accompanying table. The parameters for the particular range of $\alpha$ and $\beta$ and $\lambda$, as taken from the diagram Fig. 17 must be multiplied by these factors in order to obtain the width $w$ for optimum resolution in any special case. The diagram shows that the width becomes more and more restricted as the angle $\beta$ increases. The wave-length range, for which a satisfactory grating width can be used and a useful resolving power obtained, grows with increasing angle of incidence—or for $\beta$ nearly constant, with increasing values of $\lambda$ and of $m$. This fact is emphasized by the historical use of the stigmatic mounting: the outstanding results obtained by it were the measurements in the photographic infra-red by Meggers and his collaborators at wave-lengths, which according to Fig. 16 appear at large angles. When gratings with more lines per inch are ruled, the usefulness of the stigmatic mounting will be extended into the ultraviolet region, even for low orders.

TABLE II. Values of $(R^2d/2m)^{\frac{1}{3}}$ for $m=1$.

| $R$ | 1-meter | 10-foot | 21-foot | 30-foot |
|---|---|---|---|---|
| $d'=15{,}000\,\frac{\text{lines}}{\text{inch}}$ | 0.94 cm | 1.98 cm | 3.23 cm | 4.10 cm |
| $d=30{,}000\,\frac{\text{lines}}{\text{inch}}$ | 0.745 cm | 1.57 cm | 2.56 cm | 3.25 cm |

Some differences between this aberration in the stigmatic mounting and the aberration treated earlier (Section VI, part b) must be emphasized. The aberration represented by Eq. (89) is not a "spherical aberration," since it restricts the width of the ruled surface only, not the length of the grooves. Furthermore, the aberration is not symmetrical in $+w$ and $-w$ (as it was on the Rowland circle), but it changes its sign when $w$ changes sign. This unsymmetrical character of the aberration results from the fact that a third root has been evaluated. Therefore, $W$ in Eq. (89) represents the total width that is allowed. Furthermore, the aberration changes sign with $\beta$ on passing through the normal to the grating. For spectral lines on the normal itself, the aberration vanishes and $w$ may become infinitely large.

*2. The Aberration Arising from Higher Terms*

In analogy to the treatment given for the Rowland circle, the aberrations arising from the fourth power members in the series development must be discussed.

The third member in $F_2 + F_2'$ does not vanish for $\beta \neq 0$. It contributes to the aberration as follows:

$$\int_{-W/2}^{+W/2} \frac{\partial (F_2{}^{(3)} + F_2{}'^{(3)})}{\partial w} dW = \frac{W_{(3)}{}^4}{2} \times \left[ \frac{\sin^2 \alpha}{r^2} \left( \frac{\cos^2 \alpha}{r} - \frac{\cos \alpha}{R} \right) + \frac{\sin^2 \beta}{(r')^2} \left( \frac{\cos^2 \beta}{r'} - \frac{\cos \beta}{R} \right) \right] \leqslant \frac{\lambda}{4}. \quad (91)$$

By putting $r = \infty$ and substituting the value of $r'$ from Eq. (85), this equation becomes:

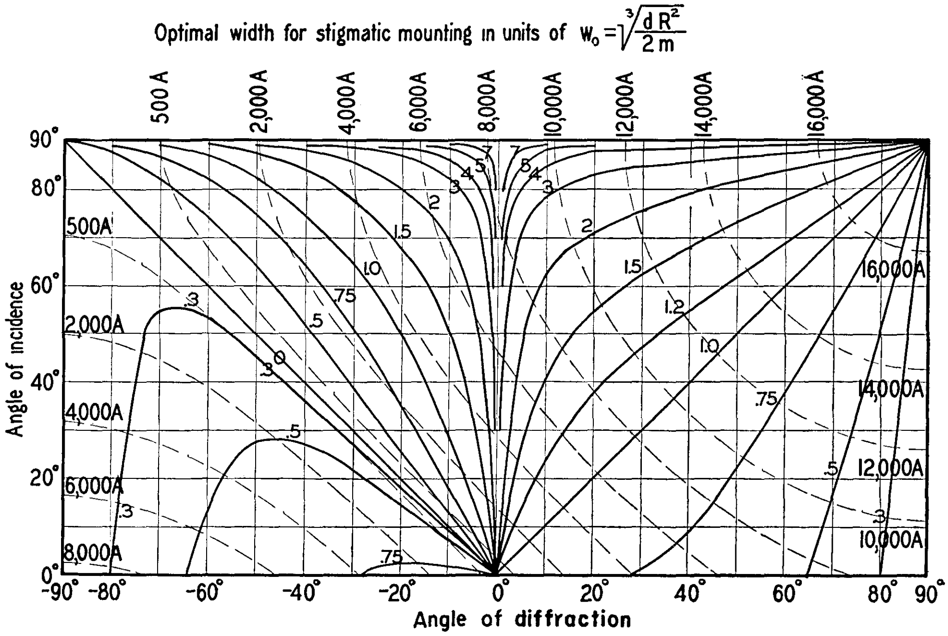


FIG. 17. Optimal width of grating in the Wadsworth mounting in units of $(dR^2/2m)^{\frac{1}{3}}$ for all angles of incidence and diffraction. Grating space, $d$; order, $m$.

$$\int_{-W/2}^{+W/2}\frac{\partial(F_2^{(3)}+F_2'^{(3)})}{\partial w}dw=\frac{W_{(3)}{}^4}{2}\cdot\frac{\sin^2\beta(\cos\alpha+\cos\beta)^2}{R^2\cdot\cos^4\beta}\times\left(\frac{\cos^2\beta(\cos\alpha+\cos\beta)}{R\cdot\cos^2\beta}-\frac{\cos\beta}{R}\right)\geqslant\frac{\lambda}{4}.$$

Hence:

$$\frac{W_{(3)}{}^4}{2}\cdot\frac{(\cos\alpha+\cos\beta)^2}{\cos^4\beta}\cdot\frac{\cos\alpha\cdot\sin^2\beta}{R^3}\leqslant\frac{\lambda}{4}. \quad (92)$$

Using the general grating Eq. (14) to eliminate $\lambda$, and introducing $(\sin\alpha+\sin\beta)/(\cos\alpha+\cos\beta)=\tan(\alpha+\beta)/2$, the solution for $W$ is:

$$W_{(3)}\leqslant 2\left(\frac{R^3d}{2m}\tan\frac{\alpha+\beta}{2}\cdot\frac{\cos^4\beta}{\cos\alpha\cdot\sin^2\beta(\cos\alpha+\cos\beta)}\right)^{\frac{1}{4}}. \quad (93)$$

The factor two before the radical will be explained later.

The value $W_{(3)}$ becomes infinite for $\beta=0$ and with a higher order than did $W$ in Eq. (89), since $\sin\beta$ to a higher order appears in the denominator. The sign of the aberration $W_{(3)}$, however, does not change if $\beta$ passes through 0°; furthermore the aberration is symmetrical in $W$, since in Eq. (91) $W$ appears in its fourth power. Therefore, in Eq. (93) a factor of two has been introduced in order to designate the allowed total width.

The numerical relation of $W_{(3)}$ to $W$ of Eq. (89) must be discussed. The factors in $R$, $d$, and $m$ yield the following ratio:

$$(R^3d/2m)^{\frac{1}{4}}:(R^2d/2m)^{\frac{1}{3}}.$$

In Table I an expression has been evaluated, that is larger by the factor $(2)^{\frac{1}{2}}$ than $(R^3d/2m)^{\frac{1}{4}}$; the values of that table therefore must be divided by 1.41. The cube root has been tabulated in Table II. The ratios of the corresponding values in the two tables are in the neighborhood of five, and after the division by 1.41, the factor of three is left in favor of the fourth root values.

The comparison of the two angular functions in Eqs. (90) and (93) can be facilitated by transformation of Eq. (93) into an expression that contains the already evaluated function of Eq. (90) as a factor, as follows:

$$W_{(3)}\leqslant 2\left[\frac{R^3d}{2m}\cdot\tan\frac{\alpha+\beta}{2}\cdot\frac{\cos^2\beta}{\cos\alpha\cdot\sin\beta}\cdot\frac{\cos\beta}{\sin\beta\left(1+\dfrac{\cos\alpha}{\cos\beta}\right)}\right]^{\frac{1}{4}}. \quad (94)$$

The last fractional factor is the only difference between the two angular functions. The magnitude of that fraction depends on the values of $\alpha$ and $\beta$. Their choice is however restricted—at least to the values for which $y_s$ [Eq. (90) and Fig. 16] is two or larger, and in addition $\alpha$ will be restricted to values smaller than 80°. The fraction $\cos\beta/\sin\beta$ will then have values larger than one, probably around three. The angle $\alpha$ will be larger than $\beta$; the total value of the last factor will probably be larger than one, but may in extreme regions decrease to one-half. Since $y_s$ was restricted to values larger than two, and $y_s$ is the cube root of the angular function, the two roots may be compared as follows:

$$f(W_{(3)}):f(W)=2(8\cdot 1/2)^{\frac{1}{4}}:(8)^{\frac{1}{3}}=2.8:2. \quad (95)$$

The factor two before the fourth root takes care of the same factor in Eq. (94). The result is, that the angular factors are about equal, even in unfavorable regions of $\alpha$ and $\beta$. A more thorough investigation shows that there is on the average a factor of two in favor of the fourth root function, especially within the regions of small aberration which may be determined from Fig. 16.

The total result is that for the usual choice of $R$ and $d$ the aberration arising from $F_2^{(3)}+F_2'^{(3)}$ is much smaller than that from the cubic members (Eq. (88)), allowing the use of three times or more the width of the grating than would be possible under the limitation of $W_{(3)}$ only. Since $W_{(3)}$ is a symmetrical aberration, it does not restrict the width allowed by Eq. (89) at all, but only shifts its range with respect to the zero point of the $w$-coordinate.

Proceeding to the other members in the series development, the quartic terms (10e, 11e) $F_5+F_5'$, which give rise to a spherical aberration must be considered, since $w$ and $l$ enter symmetrically.[35] Since these are fourth power terms, the aberration arising from them will also be symmetrical with respect to the coordinate $w$, and will therefore not restrict the value of Eq. (89) obtained from the cubic term if the magnitude of the latter is considerably smaller.

For the evaluation, the conditions $r=\infty$ and $r'=R\cos^2\beta/(\cos\alpha+\cos\beta)$ must be introduced into the Eq. (24) combined with (27).

Using the grating Eq. (14) to eliminate $\lambda$ and putting $\rho^2=w^2+l^2$, Eq. (24) becomes:

$$\frac{\rho^4}{8R^2}\left(\frac{\cos\alpha+\cos\beta}{R\cdot\cos^2\beta}-\frac{\cos\alpha+\cos\beta}{R}\right) \leqslant \frac{d}{4m}(\sin\alpha+\sin\beta). \quad (96)$$

Hence:

$$\rho^4 \leqslant \frac{2R^3\cdot d}{m}\frac{(\sin\alpha+\sin\beta)}{(\cos\alpha+\cos\beta)\cdot\tan^2\beta},$$
$$\rho \leqslant \left(\frac{2R^3\cdot d}{m}\cdot\tan\frac{\alpha+\beta}{2}\cdot\cot^2\beta\right)^{\frac{1}{4}}. \quad (97)$$

The circular aperture, with the radius $\rho$, that is allowed by Eq. (96) becomes infinite for $\beta=0$. Its magnitude for finite values of $\beta$ has to be compared with the limitations for $W$ as given by Eq. (89), where the definition of $\rho$ as one-half of the total width allowed has to be kept in mind. The comparison of the numerical factors $(2R^3d/m)^{\frac{1}{4}}$ and $(R^2d/m)^{\frac{1}{3}}$ is facilitated by Tables I and II, showing a factor of five in favor of $\rho$ in Eq. (96). The angular function can be estimated by rewriting the factor of Eq. (96) so that it contains the function of Eq. (90) evaluated in Fig. 17, as follows:

$$\left(\tan\frac{\alpha+\beta}{2}\cdot\cot^2\beta\right)^{\frac{1}{4}} = \left(\tan\frac{\alpha+\beta}{2}\cdot\frac{\cos^2\beta}{\cos\alpha\cdot\sin\beta}\cdot\frac{\cos\alpha}{\sin\beta}\right)^{\frac{1}{4}}. \quad (98)$$

The magnitude of the additional factor $\cos\alpha/\sin\beta$ must be discussed. This factor is always larger than one for the useful range of the stigmatic mounting and in the neighborhood of the normal it becomes as large as five or ten. The discussion can be restricted to the case that $y>2$ (from Eq. (90) and Fig. 17). Introducing the factor two (for $2\rho=W$) into this equation, the relation (95) becomes:

$$f(2\rho):f(W)=2(8.1)^{\frac{1}{4}}:(8)^{\frac{1}{3}}.$$

This yields a factor of at least 1.6 in favor of the fourth root, and combining it with the

[35] Compare the discussion of the aberration for the Rowland circle, and Eq. (27), Section VI, part b.

factor 5 as obtained earlier by comparing the dimensional factors, the total product becomes a width $2\rho$ that is at least eight times larger than $w$. Remembering the symmetrical character of the quartic aberration in distinction to the unsymmetrical one of the cubic aberration, it follows that the quartic expression of Eq. (96) is negligible in comparison to that of Eq. (89) and even that of Eq. (93).

### *3. Limitation of the Attainable Resolving Power by the Aberration, if the Light Striking the Grating is Strictly Parallel*

The quantitative evaluation of the aberration gave the result that at the normal itself the grating may have any size, but that even for angles as small as about $\beta = \pm 2°$ the useful width is so small, that the resolving power of the usual gratings (with 21-foot radius of curvature, 15,000 lines per inch and 6 inches ruled width) cannot be fully utilized, the optimal width at $\alpha = 40°$ to $60°$ being only four to five inches. If, however, the photographic plate used will not reveal this resolving power, the damage done by those parts of the grating that diffract the light out of phase is small. The stigmatic mounting of a grating that has the dimensions mentioned is in competition with a Rowland circle mounting with respect to resolving power only over a range of about 400A. This range is of course ample to cover the end of any Rydberg series in atoms, or an isolated band in molecular spectra.

It is evident from Eq. (89) that, the other variables being constant, an increase of the radius of curvature increases the maximum resolving power proportionally to $R^{\frac{2}{3}}$, since the allowed width grows with this power. For the same grating ($R$ and $d$ being constant) the resolving power increases with higher orders of the spectrum. If the grating has sufficient size, then the allowed width decreases with $(1/m)^{\frac{1}{3}}$, and since the resolving power increases proportionally to $m$ for any given constant width, the attainable resolving power in different orders increases with $m^{\frac{2}{3}}$, other things being equal. The comparison of gratings with the same radius of curvature but with different grating constants shows that the allowed width decreases with $(d)^{\frac{1}{3}}$; a grating with twice as many lines allows at the same angles a width that is smaller by $(\frac{1}{2})^{\frac{1}{3}}$, since the resolving power per unit width is doubled, the net gain amounts to the factor $(4)^{\frac{1}{3}} = 1.58$. But, of course, the wave-length range over which the high resolution is obtained, will be reduced by a factor of $2(2)^{\frac{1}{3}} = 2.52$, if in both cases the same fraction of the theoretical resolving power is wanted.

### *4. Dependence of the Aberration on the Collimating System*

All these considerations are rigorous only if exactly parallel light strikes the grating. This condition requires either a perfect telescope lens as a collimator, an off axis paraboloidal mirror with the slit at its focus,[36] or a combination of paraboloidal mirror and plane mirror as used by Poritzky.[37] Any spherical mirror with the slit in a "focus" off axis will show aberrations in the "parallelized" beam of light which will be greater the larger the aperture is.

The question arises if it is possible to compensate for the aberrations of the grating by those of the collimating mirror. This problem has not yet been investigated; it is however accessible by the powerful methods that have been developed in the general grating theory. The Eqs. (1)–(12) are of course rigorously valid for the concave mirror, if the grating constant $d$ is taken equal to infinity. The main consequence of this condition occurs in Eq. (14), which becomes, for $d = \infty$:

$$\sin \alpha + \sin \beta = 0. \tag{99}$$

Equation (99) is simply the law of geometrical reflection, to wit

$$\alpha = -\beta. \tag{100}$$

This relation allows the evaluation of the aberration in the light beam that is made parallel by the concave mirror. The law of the reversibility of the direction of light can be used to apply literally the formulas developed earlier; instead of making the light emerging from the slit parallel, the reverse process may be used and the parallel light focused by the mirror on the slit. This corresponds to a simple interchange of $r$, $\alpha$

[36] This was nearly realized by Fabry and Buisson, reference 31.

[37] A. Poritzky, reference 33.

and $r'$, $\beta$ in the formulas developed, and is perfectly permissible.

The collimating mirror corresponds then to a grating in Wadsworth's mounting, with $r=\infty$ and $r'=R\cdot\cos^2\beta/(\cos\alpha+\cos\beta')$ according to Eq. (85). The introduction of Eq. (100) yields the equation:

$$r'=(R/2)\cos\beta. \tag{101}$$

The main aberrations in this concave mirror are represented by the cubic terms in Eq. (23), that gave as the allowed optimum width:

$$\frac{W^3}{2}\left(\frac{\sin\beta(\cos\alpha+\cos\beta)\cdot\cos\alpha}{R^2\cdot\cos^2\beta}\right)\leqslant\frac{\lambda}{4}. \tag{87}$$

By introducing Eq. (100), this equation is simplified:

$$\frac{W^3}{2}\left(\frac{-\sin\alpha\cdot 2}{R^2}\right)\leqslant\frac{\lambda}{4}. \tag{102}$$

For the allowed width, the solution in $W$ is:

$$W\leqslant\left(\frac{R^2\cdot\lambda}{4(-\sin\alpha)}\right)^{\frac{1}{3}}. \tag{103}$$

The aberration is unsymmetric in $w$, by the same reasoning given earlier, that means that the signs of the phase shift are opposite for $-w$ and $+w$. The evaluation of the allowed width will be carried out for a 21-foot mirror at the wave-length 6000A; the slit and the grating may be close together. For a distance between them of 16 cm, $\alpha=8$ cm/320 cm$=1/40$. The calculation is as follows:

$$W\leqslant\left(\frac{640^2\cdot 6\cdot 10^{-5}}{4\cdot(1/40)}\right)^{\frac{1}{3}}=(246)^{\frac{1}{3}}=6.25\text{ cm}.$$

That is, at 6.25 cm from the center of the mirror the phase shift amounts to $\lambda/4$ or $-\lambda/4$ of the wave-length 6000A. Assume that the parallel beam from the mirror is such that rays travel from the center of the mirror to the center of the grating, that the grating has a 21-foot radius and 30,000 lines/inch; the region about 6000A appears about at the normal when, as Fig. 2 shows, an angle $\alpha$ of about 45° is subtended. From Table II, the dimensional constant for the grating used can be seen to be 2.56 cm, and in Fig. 17 the angular function for $\beta=4°$ has the value of about 2.5, so that the width allowed for the grating will be 6.4 cm, if it is illuminated by strictly parallel light. Thus at a distance of 6.4 cm from the center of the grating, diffracted light of a wave-length around 6000A has a phase shift of $\pm\lambda/4$, the sign depending on the direction of $w$.

The distance 6.25 cm at the mirror, the result from the preceding paragraph, projects itself on the grating as $6.25/\cos\alpha$, which (for $\alpha=45°$) is 8.8 cm. This example shows that the same kind of aberration occurs in the collimating concave mirror and in the concave grating, and that both have about the same order of magnitude. It is however possible to arrange the optical parts so that the two aberrations will cancel each other totally, or at least partially.

In order to find the conditions for this cancellation, it is necessary to develop the formula for mirror and grating in the same coordinate system.

Without going into the details of the derivation it may be said that if the slit and mirror are on one side of the grating normal while the spectrum is formed on the other side, as in the usual mounting, the aberrations are less than if the spectrum and mirror positions are interchanged, so that the light beams from the slit to the mirror and from the grating to the spectrum cross. In the former case the phase shifts introduced by the mirror and by the grating are in opposite senses, while in the latter case they have the same signs and a much smaller grating is required to avoid loss of resolution.

The choice of the magnitude of the angle $\alpha'$, which the slit beam makes with the axis of the mirror, and of the radius of curvature $R'$ of the mirror, offer the possibility of counterbalancing the "cubic" aberration of Eqs. (20) and (87) to a high degree for any given wave-length and order. It is only necessary to read the aberration factors from Fig. 17 and Table II, and to introduce their product $w$ into Eq. (103). The evaluation for any wanted wave-length region $\lambda$ can be carried out for a variation of either $\alpha$ or $R$. The change of $\alpha$ will probably mean a sideways motion of the slit, that of $R$ the choice of another mirror.

There is some experimental proof for the derivations in the last paragraphs. Czerny and Turner[38] experimented with an infra-red concave

[38] M. Czerny and A. F. Turner, Zeits. f. Physik **61**, 792 (1930).

mirror system corresponding to each of the arrangements mentioned and took photographs of a slit on a plate without using a grating. They found that the slit image was by far better defined, in the case where the beams did not cross, but did not give any calculations or theory. As can be seen from their pictures, the arrangement shows not only aberration, but also much astigmatism and considerable coma. However, the mathematical treatment of coma is related to that of the "cubic" aberration, and therefore the pictures of Czerny are evidence in favor of the convenional form of Wadsworth mounting.

It is apparently not possible to compensate the quartic members of Eqs. (92) and (97) by either relative position of the mirror and grating. This arises from the fact that $\sin\alpha$ and $\sin\beta$ appear only in even powers in both equations if one sets $d=\infty$ as is necessary for the collimating mirror. But as shown before, the phase shifts due to the quartic members are much smaller. The aberration arising from Eq. (96) is about one-third of that appearing on the Rowland circle.

The fact that the cubic aberration can be thus counterbalanced makes the use of paraboloidal collimating mirrors hardly necessary. Even the application of a second plane mirror between spherical collimating mirror and concave grating seems unwarranted. The same situation, of course, holds for the use of a telescope lens as a collimator (Wadsworth). The cubic aberrations of the concave grating, if illuminated by parallel light, damage the resolution so much, that it is better not to use a strictly parallel beam, but rather one that shows the phase shifts in the opposite sense to those introduced by the grating. Perfectly parallelized light is advantageous for illuminating a plane grating, but not for a concave grating.

## (c) The Astigmatism of the Wadsworth Mounting

The Wadsworth mounting is generally known as the stigmatic mounting of the grating and has been used to obtain on the plate intensity patterns that had been imaged on the slit. The stigmatism of this mounting is however much more restricted than that of a system using lenses and a plane grating or a prism. The quantitative evaluation of the astigmatism can easily be performed.

The directions of the light paths in the vertical sheets of light that include $z$ and $z'$ and $l$ at the grating, and that are inclined at the angles $\alpha$ and $\beta$ to the normal (Fig. 1) are governed by the members $F_3$ and $F_3'$ of our general formula, as shown above in Eq. (35). The application of Fermat's theorem along the coordinate $l$ yields the focusing condition for horizontal cross hairs derived in Eqs. (36)–(38) which is

$$\left(\frac{1}{r}\cos\alpha+\frac{1}{r'}-\frac{\cos\beta}{R}\right)=0. \qquad (38)$$

If parallel light strikes the grating ($r=\infty$), Eq. (38) yields for the secondary focus $r_s'$:

$$r_s'=R/(\cos\alpha+\cos\beta). \qquad (104)$$

The relations of the two focal curves, $r'$ (Eq. (85)) for the spectral lines and $r_s'$ for lines perpendicular to them, become especially simple for the case $\beta=90°$ when both reduce to

$$r'=R/(1+\cos\alpha). \qquad (105)$$

In other words: On the normal of the grating the Wadsworth mounting images the slit stigmatically. The locus for these stigmatic images is a parabola with the grating center as its focus and $R/2$ as its parameter (Eq. (19)) as has been shown earlier.

For points off the normal there is a deviation between the focal curves for $r'$ and for $r_s'$. The distance between them can easily be determined from Eqs. (85) and (104)

$$r_s'-r'=\frac{R}{\cos\alpha+\cos\beta}(1-\cos^2\beta)$$
$$=\frac{R\cdot\sin^2\beta}{\cos\alpha+\cos\beta}. \qquad (106)$$

It is clear that the secondary focal curve always lies outside the focal curve for the spectrum and that the distance between them increases symmetrically on both sides of the normal proportionally to $\sin^2\beta$ for any given angle $\alpha$. The relative deviation of the two focal curves $(r_s'-r')/r'$ from the spectral focus is especially simple:

$$\frac{r_s'-r'}{r'}=\frac{R\cdot\sin^2\beta}{\cos\alpha+\cos\beta}:\frac{R\cdot\cos^2\beta}{\cos\alpha+\cos\beta}=\tan^2\beta. \qquad (107)$$

This equation shows that the fraction $(r_s'-r')/r'$ is independent of the radius of curvature and of the angle of incidence.[39] Correspondingly, the relative deviation from the secondary focus is:

$$\frac{r_s'-r'}{r_s'}=\frac{R\cdot\sin^2\beta}{\cos\alpha+\cos\beta}:\frac{R}{\cos\alpha+\cos\beta}=\sin^2\beta. \quad (108)$$

This formula is useful in the evaluation of the length of the astigmatic images in the neighborhood of the normal. For example for $\beta=5°\ 45'$, the value of $\sin^2\beta$ is 0.01 and the spectral lines, by similar triangles, have a length of 1 percent of the length of the grating rulings. For $\beta=11°\ 20'$ the value of $\sin^2\beta$ becomes 0.039. The range of $11°\ 20'$ for a 30,000 line grating covers a wavelength range near the normal of 3000A in the first order. It is quite feasible to use such a range, and at its extreme end a point source on the slit would be drawn out to a 3-mm image for 75-cm rulings. For a 6000A range, $\sin^2 21°\ 15'=0.1320$ and the astigmatic length is 10.0 mm. Comparison with Fig. 7 shows that for the Rowland circle mounting an astigmatism as small as this is obtainable only for wave-lengths up to 4000A in the first order.

It was noted earlier that the spectral focus curves are a family of hyperbolic lemniscates. The curves for the secondary foci are a family of hyperbolas with the polar equation $r_s'=R/(\cos\alpha+\cos\beta')$, the focus of which lies in the center of the grating and the vertex of which envelops the conjugated lemniscate for $\beta=0$ and $r'=R/(\cos\alpha+1)$, which is a common tangential point of both. For $\alpha=90°$, the hyperbola degenerates into a straight line at the distance $R$ from the grating, and is tangent to the circular spectral focus curve $R\cdot\cos\beta$.

[39] Since the grating acts as a mirror for $\alpha=-\beta$, Eq. (107) gives a very simple formula for the astigmatism of a concave mirror if struck by parallel light.

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# Tests for the Detection and Analysis of Color-Blindness

## II. The Ishihara Test: Comparison of Editions

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IN an earlier report on the Ishihara test for color blindness[1] we presented a brief history of the test from the original edition, issued in 1917, to the last Japanese edition of which we have a record in this country, the ninth edition issued in 1940. Mention was also made of the British reprint of the ninth edition, issued in London in 1943.

That report is the first of a series designed to evaluate in turn various color tests as means of analyzing and detecting defective color vision. The data reported were collected during the course of a study of some 106 persons having defective color vision of varying types and amounts, including 74 who had definite color defects and 32 who had low, but normal, color vision. To study these persons a comprehensive battery of color tests was used, some of which are well-known in the field of color blindness testing, some are less well-known, and some were devised in this laboratory. The fifth edition of the Ishihara test, one of the best known of the shorter editions, was selected for inclusion in the battery of tests. The results obtained from the entire battery aided in the classification of the color defect as to type and extent and at the same time permitted us to evaluate each test of the

[1] L. H. Hardy, G. Rand, and M. C. Rittler, Tests for the Detection and Analysis of Color Blindness. I. The Ishihara Test: An Evaluation, J. Opt. Soc. Am. **35**, 268 (1945).