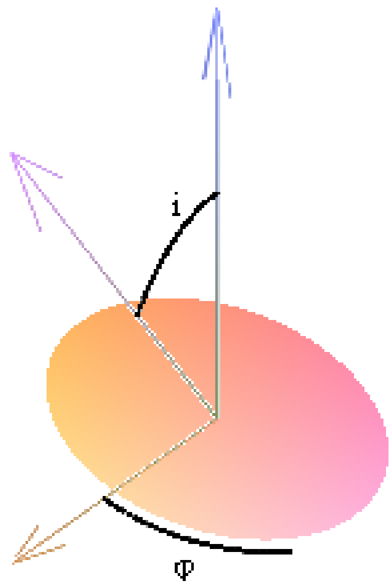
The intensity of a radiating gaseous disk as a function of the distance from its central star

Abstract

Stars with a gaseous disk show emission lines e. g. Hα. This paper shows how a spectrum of an emission line can be used to calculate the distribution of the intensity of the radiation depending on the distance from the star. This calculation can be done in the case of keplerian rotation of the disk or pure expansion without rotation.

Rotating disk

Be stars are characterized by the presence of Balmer emission lines due to some circumstellar gaseous disk. In this paper the particles of the disk shall travel in circular Keplerian orbits and the intensity of radiation shall depend on the distance from the star only. Furthermore we assume that the given emission line has no blends with other emission or absorption lines and that the thermal broadening of this line is very small compared to its equivalent width. We define polar coordinates (r, ϕ) in the central plane of the disk with the star M as the center. The view ray of the observer shall cross the radial axis under the angle i. Let Pr be the projection of the view ray on the central plane of the disk. The projection of a velocity vector of a given particle is called v_{pr}. The projection of the velocity vector on the view ray is $v_{Ob} = v_{pr} * \sin i$. $\phi = 0$ characterizes the ray perpendicular to Pr, so that the rays $\phi = \pi/s$ and $\phi = 3\pi/s$ are parallel to Pr. The we have



Gaseous disk

$v_{pr}^2 = v^2\cos^2\varphi = G\cdot M\left(\frac{2}{r} - \frac{1}{a}\right)$ with a = semiminor axis of kepler-ellipse. Because of the circular orbit we have r= a:

$$v_{pr} = v*\cos\phi = \sqrt{\frac{GM}{r}}\ \cos\varphi \qquad (1)$$

The Isotache for a given v_{pr} is determined by

$$\varphi = \arccos\left(v_{pr}\cdot\sqrt{\frac{r}{G\cdot M}}\right) \quad \text{or} \quad r = \frac{G\,M}{v_{pr}^{\ 2}}\cos^2\varphi \qquad (2)$$

$$\frac{d\varphi}{dr} = \frac{v_{pr}}{2\sqrt{G\cdot M\cdot r}}\ \frac{1}{\sqrt{1 - \frac{v_{pr}^{\ 2}\cdot r}{G\cdot M}}} \qquad (3)$$

since $\quad v_{pr}\sqrt{\frac{r}{G\cdot M}} \le v\sqrt{\frac{r}{G\cdot M}} = 1$

The area in polar coordinates is given by the well known formula

$$F = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2\, d\varphi \qquad (4)$$

All point with a given v_{pr} are located on a closed curve called isotach. All points in the interior belong to velocities larger than v_{pr}.

$$F(v > v_{pr})$$

The area between the isotachs for v_{pr} and $v_{pr} + dv_{pr}$ is

$$F(v_{pr} \le v \le v_{pr} + dv_{pr}) = -\frac{G^2\cdot M^2}{2}\int_{\varphi_1}^{\varphi_2}\left[\frac{1}{v_{pr}^{\,4}} - \frac{1}{(v_{pr} + dv_{pr})^4}\right]\cos^4\varphi\, d\varphi \qquad (5)$$

with $\varphi_1 = -\frac{\pi}{2}$ and $\varphi 2 = \frac{\pi}{2}$. For $\phi = 0$ is $v_{pr} = v$. We obtain

$$\frac{dF}{dv_{Ob}} = 2\, G^2M^2\int_{\varphi_1}^{\varphi_2}\frac{\cos^4\varphi}{v_{pr}^{\,5}}\, d\varphi =$$

$$= 2\, G^2M^2\int_0^{\frac{GM}{v_{pr}^{\,2}}}\frac{1}{v_{pr}^{\,5}}\left(\frac{v_{pr}^{\,2}\, r}{GM}\right)^2\frac{v_{pr}}{2\sqrt{GMr}}\,\frac{1}{\sqrt{1 - \frac{v_{pr}^{\,2}\cdot r}{G\cdot M}}}\, dr$$

$$= \frac{2}{\sqrt{GM}}\int_0^{\frac{GM}{v_{pr}^{\,2}}}\frac{r^{\frac{3}{2}}}{\sqrt{1 - \frac{v_{pr}^{\,2}\cdot r}{G\cdot M}}}\, dr \qquad (6)$$

An additional factor 2 is due to the fact that the substitution for $-\pi/2 \le \phi \le 0$ and $0 \le \phi \le \pi/2$ has to be accomplished for each case seperately. Both results are equal and must be added.

With the intensity f(r) and $\Delta\lambda = -\frac{v_{Ob}}{c}\lambda_0$ yields for the spectrum

$$g(\Delta\lambda) = \frac{2}{\sqrt{GM}} \int_0^{\frac{GM}{v_{pr}^2}} \frac{r^{\frac{3}{2}} f(r)}{\sqrt{1 - \frac{v_{pr}^2 \cdot r}{G \cdot M}}} \, dr = \frac{2}{\sqrt{GM}\, v_{pr}} \int_0^{\frac{GM}{v_{pr}^2}} \frac{r^{\frac{3}{2}} f(r)}{\sqrt{\frac{1}{v_{pr}^2} - \frac{r}{G \cdot M}}} \, dr \qquad (7)$$

where $g(\Delta\lambda)$ will be determined by measurements. As an example this yields for the homogeneous density f(r) =1 [3]

$$g(\Delta\lambda) = \frac{3}{4} \frac{\pi \cdot G^2 M^2}{v_{pr}^5} = \frac{3\, G^2 M^2}{4\,(c\,\Delta\lambda)^5} \left(\lambda_0 \sin i\right)^5$$

To yield a normalized form of (7) we apply some transformations. We define

$$\xi = \frac{1}{v_{pr}^2}\;,\; \varrho = \frac{r}{GM},\; \mathbf{f}(\varrho) = 2\, G^2 M^2 \varrho^{\frac{3}{2}} f(GM\varrho) \text{ and}$$

$$\mathbf{g}(\xi) = v_{pr}\, g(\Delta\lambda) = v_{pr}\, g\left(-\frac{v_{pr} \sin i}{c} \lambda_0\right) = \frac{1}{\sqrt{\xi}}\, g\left(-\frac{\sin i\, \lambda 0}{c\sqrt{\xi}}\right) \qquad (8)$$

The first definition implicates a distinction between two cases. We must decide, whether we want to treat the redshift or the blueshift part of the emission line. For the upper Integration limit is $r = \frac{GM}{v_{pr}^2} \Rightarrow \varrho = \xi$. From (7) we conclude because of $dr = GM\,d\varrho$

$$\mathbf{g}(\xi) = \frac{G^{\frac{3}{2}} M^{\frac{3}{2}}}{\sqrt{GM}} GM \int_0^{\xi} \frac{\varrho^{\frac{3}{2}} f(GM\varrho)}{\sqrt{\xi - \varrho}} \, d\varrho = G^2 M^2 \int_0^{\xi} \frac{\varrho^{\frac{3}{2}} f(GM\varrho)}{\sqrt{\xi - \varrho}} \, d\varrho = \int_0^{\xi} \frac{\mathbf{f}(\varrho)}{\sqrt{\xi - \varrho}} \, d\varrho \qquad (9)$$

This is an Abelian integral equation for the unknown function f(ϱ) with the solution [1],[2]

$$\mathbf{f}(\varrho) = \frac{1}{\pi} \frac{d}{d\rho} \int_0^{\varrho} \frac{\mathbf{g}(\xi)}{\sqrt{\varrho - \xi}} \, d\xi \qquad (10)$$

Entering (8) into (10) yields

$$f(r)=\frac{1}{2\pi\sqrt{GM}r^{\frac{3}{2}}}\frac{dr}{d\varrho}\frac{d}{dr}\int_0^{\frac{r}{GM}}\frac{g\left(-\frac{\lambda_0\sin i}{c\sqrt{\xi}}\right)}{\sqrt{\xi}\sqrt{\frac{r}{GM}-\xi}}\,\mathrm{d}\xi=\frac{\sqrt{GM}}{2\pi r^{\frac{3}{2}}}\frac{d}{dr}\int_0^{\frac{r}{GM}}\frac{g\left(-\frac{\lambda_0\sin i}{c\sqrt{\xi}}\right)}{\sqrt{\xi}\sqrt{\frac{r}{GM}-\xi}}d\xi$$

(11)

However it is more comfortable to keep the variable ϱ which is equivalent to GM as lenght unit, even GM has not the unit m.
Substitute η by ξ with η = 1/ξ :

$$f(\varrho)=\frac{1}{2\pi G^2M^2\varrho^{\frac{3}{2}}}\frac{d}{d\varrho}\int_0^{\varrho}\frac{g\left(-\frac{\lambda 0\sin i}{c\sqrt{\xi}}\right)}{\sqrt{\xi}\sqrt{\varrho-\xi}}\,\mathrm{d}\xi=\frac{1}{2\pi G^2M^2\varrho^{\frac{3}{2}}}\frac{d}{d\varrho}\int_{\frac{1}{\varrho}}^{\infty}\sqrt{\eta}\frac{g\left(-\frac{\lambda 0\sin i}{c}\sqrt{\eta}\right)}{\sqrt{\varrho-\frac{1}{\eta}}}\frac{1}{\eta^2}\,\mathrm{d}\eta$$

$$=\frac{1}{2\pi G^2M^2\varrho^{\frac{3}{2}}}\frac{d}{d\varrho}\int_{\frac{1}{\varrho}}^{\infty}\frac{1}{\sqrt{\varrho}}\frac{g\left(-\frac{\lambda 0\sin i}{c}\sqrt{\eta}\right)}{\eta\sqrt{\eta-\frac{1}{\varrho}}}\,\mathrm{d}\eta=\frac{1}{2\pi G^2M^2\varrho^{\frac{3}{2}}}\frac{d}{d\varrho}\left[\frac{1}{\sqrt{\varrho}}\int_{\frac{1}{\varrho}}^{\infty}\frac{g\left(-\frac{\lambda 0\sin i}{c}\sqrt{\eta}\right)}{\eta\sqrt{\eta-\frac{1}{\varrho}}}\,\mathrm{d}\eta\right]$$

$$=\frac{-1}{4\pi G^2M^2\varrho^3}\int_{\frac{1}{\varrho}}^{\infty}\frac{g\left(-\frac{\lambda 0\sin i}{c}\sqrt{\eta}\right)}{\eta\sqrt{\eta-\frac{1}{\varrho}}}\,\mathrm{d}\eta+\frac{1}{2\pi G^2M^2\varrho^2}\frac{d}{d\varrho}\int_{\frac{1}{\varrho}}^{\infty}\frac{g\left(-\frac{\lambda 0\sin i}{c}\sqrt{\eta}\right)}{\eta\sqrt{\eta-\frac{1}{\varrho}}}\,\mathrm{d}\eta$$

(12)

Pure expansion

The method of calculation is the same as we used for the circular case. Let v(r) denominate the velocity at the distance r from the star we have

$$\mathrm{v}_{pr}=\sin\varphi\;\mathrm{v}(r)=const\quad\text{or}\quad\varphi=\arcsin\left(\frac{\mathrm{v}_{pr}}{\mathrm{v}(r)}\right)\text{ and}\qquad(13)$$

$$r = \mathrm{v}^{-1}\left(\frac{\mathrm{v_{pr}}}{\sin\varphi}\right) \qquad (14)$$

Because the cos is replaced by sin the isotachs are rotated by π/2. When $\phi \rightarrow 0$ then $r \rightarrow 0$ and for $\varphi = \frac{\pi}{2}$ is $r = \mathrm{v}^{-1}(\mathrm{v_{pr}})$

We can use (4) again and get

$$F(\mathrm{v} > \mathrm{v_{pr}}) = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2\, d\varphi = \frac{1}{2}\int_{\varphi_1}^{\varphi_2}\left(\mathrm{v}^{-1}\left(\frac{\mathrm{v_{pr}}}{\sin\varphi}\right)\right)^2 d\varphi \qquad (15)$$

With $D\mathrm{v}^{-1}$ the derivative of the inverse of v(r) we have

$$\frac{dF}{d\mathrm{v}_{pr}} = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} 2\,\mathrm{v}^{-1}\left(\frac{\mathrm{v_{pr}}}{\sin\varphi}\right) D\mathrm{v}^{-1}\left(\frac{\mathrm{v_{pr}}}{\sin\varphi}\right)\frac{1}{\sin\varphi}\, d\varphi$$

$$=-\int_{r_1}^{r_2} \mathrm{r}\; D\mathrm{v}^{-1}(\mathrm{v(r)})\;\frac{\mathrm{v}(r)}{\mathrm{v_{pr}}}\;\frac{1}{\sqrt{1-\frac{\mathrm{v_{pr}}^2}{\mathrm{v}^2(r)}}}\;\frac{\mathrm{v_{pr}}}{\mathrm{v}^2(r)}\;\frac{d}{dr}\mathrm{v}(r)\; dr \qquad (16)$$

The r comes from the first factor in the first integral, the $\mathrm{v(r)}/\mathrm{v_{pr}}$ from $1/\sin\phi$ and the last three factors are due to the substitution $\phi \rightarrow r$. With $r_1 = 0$ and $r2 = \frac{1}{\mathrm{v_{pr}}^2}$ we get finally

$$\frac{dF}{d\mathrm{v_{pr}}} = \int_{r_1}^{r_2}\frac{\mathrm{r}}{\sqrt{\mathrm{v}^2(r) - \mathrm{v_{pr}}^2}}\; D\mathrm{v}^{-1}(\mathrm{v(r)})\;\frac{1}{\sin\varphi}\; dr \qquad (17)$$

We specialize this the case of escape velocity, i.e. $\mathrm{v}(r) = \frac{\sqrt{2GM}}{\sqrt{r}}$ and get with $a = \sqrt{2GM}$

$$\frac{dF}{d\mathrm{v_{pr}}} = \frac{1}{2}\int_{r_1}^{r_2}\frac{r\,a^2}{a\sqrt{\frac{a^2}{r} - \mathrm{v_{pr}}^2}}\;\frac{1}{\sqrt{r}^3}\,(-2a^2)\,\frac{\sqrt{r}^3}{a^3}\; dr = \frac{1}{\mathrm{v_{pr}}}\int_0^{\frac{2GM}{\mathrm{v_{pr}}^2}}\frac{\sqrt{r}^3}{\sqrt{\frac{2GM}{\mathrm{v_{pr}}^2} - r}}\; dr$$

$$= \frac{2}{\sqrt{G M}\,\mathrm{v_{pr}}} \int_0^{\frac{2 G M}{\mathrm{v_{pr}}^2}} \frac{r^{\frac{3}{2}}}{\sqrt{\frac{1}{\mathrm{v_{pr}}^2} - \frac{r}{2\, G \cdot M}}}\, \mathrm{d}r \qquad (18)$$

which is identical to (6) but G M is replaced by 2 G M. With the intensity function f(r),

$\varrho = \frac{r}{2\, G M}$, $\mathbf{f}(\varrho) = 8\, G^2 M^2 \varrho^{\frac{3}{2}} f(G M \varrho)$ and $\mathbf{g}(\xi)$ from (8) we can write again

$$\mathbf{g}(\xi) = \int_0^{\xi} \frac{\mathbf{f}(\varrho)}{\sqrt{\xi - \varrho}}\, \mathrm{d}\varrho \qquad (19)$$

$$\mathbf{f}(\varrho) = \frac{1}{\pi} \frac{d}{d\rho} \int_0^{\varrho} \frac{\mathbf{g}(\xi)}{\sqrt{\varrho - \xi}}\, \mathrm{d}\xi \qquad (20)$$

Accordingly (11) and (12) hold with 2 G M instead of G M.

Some remarks

The different kinematic behaviour of the central star is in these formulas not explicitly considered. However in most cases a star has only absorption lines and no emission lines. A sharp absorption line is broadened to an elliptical shape. High temperature and pressure may also influence its shape. When this absorption line is subtracted we should get the distribution of the emission line in the exterior of the star. With $g(\Delta\lambda)$=const the integral in (11) has the value π independently of r. Due to the differential operator the result of (11) is zero. Therefore the level of the continuum is irrelevant.

[1] E. C. Titchmarsh, Introduction to the theory of Fourier integrals, 1948
[2] W. I. Smirnow, Lehrgang der höheren Mathematik, 1963
[3] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series and products, 2007