The essentials of this paper were contained in a talk[8] delivered by the author before a meeting of the American Astronomical Society at Wellesley College on September 12, 1940.

*Summary.*—The axial rotation of a component of a spectroscopic binary combines with its tidal distortion and with its darkening (both ordinary and gravitational) to influence the component's observed radial velocity. The amount of the contribution to the radial velocity is computed, for all stellar models. It is found that even when an orbit is really circular the contribution is usually such as to lead, when not allowed for, to a non-vanishing spectroscopic eccentricity and to a longitude of periastron of 90°.

[1] Sterne, *Proc. Nat. Acad. Sci.*, **27**, 93–99 (1941).
[2] Sterne, *Proc. Nat. Acad. Sci.*, **27**, 93–106 (1941).
[3] Sterne, *Proc. Nat. Acad. Sci.*, **26**, 36 (1940).
[4] Luyten, *Pub. Minn.*, **2**, 29 (1935).
[5] Martin, *B. A. N.*, **8**, 265 (1938).
[6] Baker, *L. O. B.*, **12**, 130 (1926).
[7] Baker, *Allegheny Pub.*, **1**, 77 (1909).
[8] Sterne, *Proc. A. A. S.*, **10**, 68 (1940).

# *NOTES ON BINARY STARS. V. THE DETERMINATION BY LEAST-SQUARES OF THE ELEMENTS OF SPECTROSCOPIC BINARIES*

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*Introduction.*—In Note IV we studied[1] certain contributions, of a non-orbital nature, to the radial velocity of a component of a spectroscopic binary. We now suppose that the non-orbital part of the observed radial velocity either has been allowed for or is negligible, and we concern ourselves with the problem of determining the orbital elements from the orbital radial velocity. For such determinations the method of least-squares has advantages, in general, not possessed by any other method and it has usually been employed. Luyten, however, has recently pointed out[2] that the usual type of least-squares solution is not well adapted to orbits of very small eccentricity. Here we describe two forms of least-squares solution by the use of which one may retain all the advantages peculiar to the method of least-squares, while avoiding the difficulty mentioned by Luyten.

The fundamental and well-known equation for the radial velocity, $V$, of a component of a spectroscopic binary is:

$$V = \gamma + Ke \cos \omega + K \cos u \quad (1)$$

where $\gamma$ is the radial velocity of the binary's center of mass, and $e$ is the orbital eccentricity. The argument of the latitude, $\omega + v$, is denoted by $u$; $\omega$ is the longitude of periastron, measured from the ascending node of the component under consideration, and $v$ is the true anomaly. The half-range in velocity, $K$, is given by

$$K = \mu a \sin i/(1 - e^2)^{1/2} \quad (2)$$

where $\mu$ is the mean angular motion, $i$ is the inclination and $a$ is the mean distance of the orbit of the component relative to the binary's center of mass. The value of $V$ for any date can be computed by equation (1) if one knows $\mu$ (or $P$, the period), $K$, $e$, $\gamma$, $\omega$ and $T$ (the date of periastron passage); and it is usual to employ the preceding six quantities as the elements of the orbit. The problem in practice is to determine them from the observed values of $V$. For this purpose many methods have been devised, which fall into two classes. Some methods, such as Russell's "short" method[3] and the Wilsing-Russell method,[4] aim at direct solutions, while the method developed by Schlesinger[5] determines, by least-squares, the differential corrections to a set of preliminary elements obtained by any direct method.

To obtain a definitive orbit, there is on general grounds but little doubt that differential correction by least-squares is the best procedure. It consumes but a small fraction of the amount of time that is spent in exposing and measuring the spectrogram plates, yet it eliminates all personal bias from the computations, it allows one to weight the observations easily and correctly and it enables one to find both the set of elements that best represent the observations and the mean errors of those elements. The usual form of least-squares solution is Schlesinger's;[5] the elements that are corrected differentially are $\gamma$, $K$, $\omega$, $e$, $T$ and $\mu$; and tables have been published[5,6] to facilitate the derivation of the necessary coefficients.

When $e$ is not small, the usual type of least-squares solution encounters no difficulties. Luyten, however, has pointed out[2] that many computers, when applying Schlesinger's method to orbits of small eccentricity, have fixed $T$ or $\omega$ and have thereby obtained misleadingly small values for the mean errors of the elements. Luyten thinks that the usual form of least-squares solution is not really applicable to orbits of very small eccentricity, and that $T$ is not a suitable element, being nearly indeterminate in nearly circular orbits. He has replaced $T$ by the date of nodal passage, which remains determinate even in the limit of vanishing eccentricity, and, abandoning the method of least-squares, he has carried out numerous redeterminations[2] of the elements of spectroscopic binaries, by the Wilsing-Russell method.

The author thinks that the fixing of $T$ or $\omega$ has been a mistake of computers rather than an intrinsic defect in methods of differential correction by least-squares, but he agrees with Luyten that $T$ is not the most suitable element for introducing the time, and he proposes instead the general employment of the date, $T_0$, at which the mean longitude $\omega + M$ is zero, $M$ being the mean anomaly. One may call $T_0$ the "epoch of the mean longitude." Like Luyten's date of nodal passage, $T_0$ remains determinate in the limit of vanishing $e$; but $T_0$ lends itself somewhat more readily than Luyten's element to the computation of an ephemeris, and to differential correction by the method of least-squares. The mean longitude at date $t$ is merely $\mu(t - T_0)$. The change that is necessary in Schlesinger's least-squares procedure, to correct $T_0$ instead of $T$, is small and will be described. But when $e$ is very nearly zero, the preliminary orbit is best taken to be circular; and then the coefficients of the differential correction to $e$, in the least-squares solution, become indeterminate. Neither the Schlesinger procedure nor the modification of it that involves $T_0$ can be applied to a perfectly circular preliminary orbit. Therefore another form of least-squares solution will later be described that is suitable for orbits of such small eccentricity that the preliminary value of $e$ is zero. The two forms of least-squares solution are complementary. The first form is suitable for all orbits except those having very small $e$'s; the second form is suitable for orbits having very small $e$'s, and for no others. When it can be used, the second form involves considerably less computation than the first.

*The First Form of Least-Squares Solution.*—In Schlesinger's[5] solution, the equations of condition have the form

$$\delta V = \Gamma + \cos u \cdot \kappa + \sin u \cdot \pi + \alpha \sin u \cdot \epsilon + \beta \sin u \cdot \tau + \beta \sin u \cdot (t - T)m \quad (3)$$

where $\delta V$ is the observed radial velocity *minus* the radial velocity computed from the preliminary elements, and where

$$\begin{aligned}
\alpha &= 0.452 \sin v \cdot (2 + e \cos v), \\
\beta &= (1 + e \cos v)^2/(1 + e)^2, \\
\Gamma &= \delta\gamma + e \cos \omega \cdot \delta K + K \cos \omega \cdot \delta e - Ke \sin \omega \cdot \delta\omega, \\
\kappa &= \delta K, \\
\pi &= -K\delta\omega, \\
\epsilon &= -K\delta e/[0.452(1 - e^2)], \\
\tau &= K\mu[(1 + e)/(1 - e)^3]^{1/2}\delta T, \\
m &= -K[(1 + e)/(1 - e)^3]^{1/2}\delta\mu.
\end{aligned} \quad (4)$$

Values of the quantities $\alpha$ and $\beta$ are listed in Schlesinger's tables 1 and 3 of reference 5, while tables of $v$ as a function of $M$ and of $e$ are available in reference 6.

Now $T = T_0 + (\omega/\mu)$ where $T_0$ is the date at which the mean longitude is zero. Hence:

$$\delta T = \delta T_0 + \frac{\delta\omega}{\mu} - \frac{\omega}{\mu^2}\delta\mu,$$

and therefore

$$\tau = \tau_0 - [(1 + e)/(1 - e)^3]^{1/2}\pi + (\omega/\mu)m,$$

where

$$\tau_0 = K\mu[(1 + e)/(1 - e)^3]^{1/2}\delta T_0. \qquad (4')$$

The equations of condition for finding the differential corrections to the elements $\gamma$, $K$, $\omega$, $e$, $T_0$ and $\mu$ are therefore of the form

$$\delta V = \Gamma + \cos u \cdot \kappa + \sin u \cdot \left(1 - \beta \sqrt{\frac{1 + e}{(1 - e)^3}}\right)\pi + \alpha \sin u \cdot \epsilon + \beta \sin u \cdot \tau_0 + \beta \sin u \cdot (t - T_0)m. \qquad (5)$$

To find the differential corrections to the elements $\gamma$, $K$, $\omega$, $e$, $T_0$ and $\mu$, by least-squares, it is merely necessary to solve the equations of condition of the form (5) for the quantities $\Gamma$, $\kappa$, $\pi$, $\epsilon$, $\tau_0$ and $m$.

The equations (5) are almost the same as Schlesinger's equations (3), and the only changes are in the coefficients of $\pi$ and $m$ and in the replacing of $\tau$ by $\tau_0$. The coefficient of $\pi$ in (5) is readily evaluated as sin $u$ diminished by a constant multiple $[(1 + e)/(1 - e)^3]^{1/2}$ of the coefficient of $\tau_0$; while the coefficient of $m$ is the same as in Schlesinger's solution except that his $(t - T)$ has become $(t - T_0)$. The tables[5, 6] employed in Schlesinger's method should be used.

As an interesting example of the present least-squares method, the author has applied it to the primary component of $u$ Herculis. Baker's[7] normals were used, and the adopted preliminary elements were $\gamma = -22$ km./sec.; $K = 100$ km./sec.; $\omega = 83°$; $e = 0.05$; $T_0 = $ 1908 July $2^d.427$. The period, $2^d.0510$, is known from a longer series and was not varied. The definitive elements that were found by the least-squares solution were $\gamma = -20.9$ km./sec.; $K = 99.2 \pm 1.5$ km./sec.; $\omega = 85° \pm 16°$; $e = 0.056 \pm 0.016$; $T_0 = $ 1908 July $2^d.422 \pm 0^d.005$. The errors are mean errors. An unmodified Schlesinger-type solution yielded $T = $ 1908 July $2^d.904 \pm 0^d.092$; the other elements being of course the same for both solutions. It will be noticed how much more accurately $T_0$ is determined than $T$. Baker's solution[7] fixed $T$ at 1908 July $2^d.80$ and yielded $\gamma = -21.2$ km./sec.; $K = 99.50 \pm 0.99$ km./sec.; $\omega = 66°.15 \pm 0°.54$; $e = 0.053 \pm 0.010$ (probable errors). His probable errors and values agree sufficiently well with our mean errors and values for $K$ and for $e$; but his $\omega$ differs considerably from ours and his value of its probable error is much too small. Luyten, by the Wilsing-Russell method, finds[8] the values $\omega = 83° \pm 19°$;

$e = 0.047 \pm 0.016$; $T =$ 1908 July $2^d.898$ (mean errors); these values and their mean errors (when given) are substantially the same as ours.

As pointed out[1] in Note IV, the numerical values of $e$ and of $\omega$ are probably influenced by non-orbital effects in the observed radial velocity. We have not allowed for such effects here because we have wished to compare our results with those of Baker and Luyten, who applied no corrections; and because the exact values of the corrections could be reached only by rather lengthy discussion.

*The Second Form of Least-Squares Solution.*—If $e$ in the preliminary solution is allowed to approach zero, equation (5) becomes in the limit

$$\delta V = \delta\gamma + K\cos\omega\cdot\delta e + \cos u\cdot\delta K - 2K\sin v\cdot\sin u\cdot\delta e + K\mu\sin u\cdot\delta T_0 - K\sin u\cdot(t - T_0)\delta\mu.$$

Replace $u$ by $L$, the mean longitude; $\delta e$ by $e$ (the improved value of $e$, the preliminary value being zero); and $v$ by $L - \omega$. Then

$$\delta V = \delta\gamma + \cos L\cdot\delta K + \cos 2L\cdot Ke\cos\omega + \sin 2L\cdot Ke\sin\omega + \sin L\cdot\mu K\delta T_0 - \sin L\cdot(t - T_0)K\delta\mu. \quad (6)$$

Equation (6) is the simple form assumed by the equations of condition when the preliminary orbit is circular with the elements $\gamma$, $K$, $T_0$ and $\mu$. In (6), $\delta V$ is merely the observed radial velocity *minus* the radial velocity computed for the preliminary orbit, and the latter velocity is merely $\gamma + K\cos L$, where $L = \mu(t - T_0)$, and is thus very readily computed. A least-squares solution of the equations of condition of the form (6), with appropriate weights, will yield the values of the unknowns (in which $K$ and $\mu$ have their preliminary values) $\delta\gamma$, $\delta K$, $Ke\cos\omega$, $Ke\sin\omega$, $\mu K\delta T_0$ and $K\delta\mu$, along with their mean errors. One thus obtains definitive values and mean errors for $\gamma$, $K$, $e\cos\omega$, $e\sin\omega$, $T_0$ and $\mu$. One may regard these six quantities as the elements; or one may go further and find $e$ and $\omega$ separately from $e\cos\omega$ and $e\sin\omega$.

Denote $e\sin\omega$ by $g$ and $e\cos\omega$ by $h$. Denote the mean errors of $g$ and $h$, found from the least-squares solution, by $\sigma_g$ and by $\sigma_h$. Then if the weight of the observations is uniformly distributed along the velocity-curve, $g$ and $h$ will not be correlated in the least-squares solution and the mean error, $\sigma_e$, of $e$ is given by

$$\sigma_e^2 = (g^2\sigma_g^2 + h^2\sigma_h^2)/e^2 \quad (7)$$

and the mean error, $\sigma_\omega$, of $\omega$ is given by

$$\sigma_\omega^2 = (h^2\sigma_g^2 + g^2\sigma_h^2)/e^4. \quad (8)$$

If the weight is not uniformly distributed, there may be correlation between $g$ and $h$ which would render difficult any precise computation of the mean errors of the separated $e$ and $\omega$. A method of separating $e$ and $\omega$ and of obtaining their formally correct mean errors, under any circumstances, would be to apply a second differential correction by the first method of this Note. It is believed, however, that the mean errors found by equations (7) and (8) will be sufficiently close to the truth in most practical cases to give a good idea of the accuracies of $e$ and of $\omega$. It would require a most extraordinary distribution of the observations to render the computed $\sigma_e$ and $\sigma_\omega$ incorrect as to order of magnitude.

If the preliminary orbit is circular, and if the improved orbit as obtained by solving equations (6) should turn out to have an $e$ larger than about 0.05, it might be advisable not to regard the improved orbit as definitive, and to apply still another differential correction by the first method of this Note. The limit could be raised somewhat, perhaps to 0.10, for observations not of the highest quality; but it is the special merit of the present method that it deals satisfactorily with orbits having very small $e$'s, and it thus remedies the deficiencies of the first method. Whenever $e$ is so large as to render the present method of doubtful accuracy (because of the corrections' no longer being differentials), the first method is applicable without difficulty and is the logical method to apply—to a preliminary elliptical orbit.

As an example, the author has applied the second method of this Note to Baker's[7] normals of the primary of $u$ Herculis. The example should be of interest because the solution by the first method yielded the appreciable eccentricity of 0.056. The preliminary circular orbit, to which the second method was applied, had the elements $\gamma = -22$ km./sec.; $K = 101$ km./sec.; $T_0 = 1908$ July $2^d.418$. The period, $2^d.0510$, being known from a longer series, was not varied. The second method yielded the values $\gamma = -21.3$ km./sec.; $K = 99.9 \pm 1.4$ km./sec.; $e \cos \omega = -0.003 \pm 0.014$; $e \sin \omega = 0.053 \pm 0.014$; $T_0 = 1908$ July $2^d.423 \pm 0^d.005$. From the values of $e \cos \omega$ and $e \sin \omega$ one finds $e = 0.053 \pm 0.014$; $\omega = 93° \pm 15°$. The errors are mean errors. One sees that these elements and mean errors are in good agreement with the elements and mean errors that were yielded by the first method, which started with an elliptical preliminary orbit.

I am grateful to Professor Russell for his discussion, by correspondence, of the relative merits of some alternative spectroscopic elements.

*Summary.*—Luyten has pointed out that the usual form of least-squares solution is unsuitable to orbits of very small eccentricity. Two forms of solution are presented that enable one to avoid the difficulty described by Luyten, while retaining the advantages of least-squares. The first form is a modification of Schlesinger's method, in which the date of periastron passage, $T$, is replaced by the date, $T_0$, at which the mean longitude is zero. Schlesinger's tables can be used. The general use of $T_0$ as an element, instead of $T$, is recommended. The second form is a least-squares solution that involves $T_0$, $e \cos \omega$ and $e \sin \omega$ as elements. The first form is suitable

for all orbits except those with very small $e$'s. The second form is particularly suitable for orbits having very small $e$'s, and is unusually simple.

[1] Sterne, *Proc. Nat. Acad. Sci.*, **27**, 168–175 (1941).
[2] Luyten, *Minn. Pub.*, **2**, 53 (1936).
[3] Russell, *Ap. Jour.*, **40**, 282 (1914).
[4] Russell, *Ap. Jour.*, **15**, 252 (1902).
[5] Schlesinger, *Allegheny Pub.*, **1**, 33 (1908).
[6] Schlesinger and Udick, *Allegheny Pub.*, **2**, 155 (1912).
[7] Baker, *Allegheny Pub.*, **1**, 77 (1909).
[8] Luyten, *Minn. Pub.*, **2**, 29 (1935).

# *ON REAL CLOSED CURVES OF ORDER $n + 1$ IN PROJECTIVE $n$-SPACE*[1]

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We consider curves in real projective $n$-space $R_n$.

We call a curve differentiable at one of its points $s$ if it possesses there linear osculating spaces $L_p^n = L_p^n(s)$ of all dimensions $p(-1 \leqslant p \leqslant n)$. That is: we define $L_{-1}^n$ as the empty space. Suppose we already have defined $L_p^n$ and postulated its existence. Then we postulate that the linear $(p + 1)$-spaces through $L_p^n$ and a point $s'$ moving on the curve toward $s$ have a limit space which we call the osculating $(p + 1)$-space $L_{p+1}^n$.

If $L_{n-1}^n(s)$ has only a finite number of points in common with the curve, there exists a one row matrix $(a_0, a_1, \ldots, a_{n-1})$, the characteristic of the point $s$, such that 1° each of the numbers $a_0, a_1, \ldots, a_{n-1}$ equals 1 or 2, and 2° every $(n - 1)$-space containing exactly $L_p^n$ lies wholly on one side of a sufficiently small open partial arc containing $s$ if the sum $a_0 + a_1 + \ldots + a_p$ is even; it cuts it if this sum is odd $(p = 1, \ldots, n - 1)$.[2]

If the point $s$ has the characteristic $(a_0, a_1, \ldots, a_{n-1})$ and if a subspace contains $L_p^n(s)$ but not $L_{p+1}^n(s)$, we count $s$ as an $(a_0 + a_1 + \ldots + a_p)$-fold intersection of the curve with the subspace.

Let the sum of the multiplicities of the intersections of a differentiable curve with a $(n - 1)$-space have the maximum $m$; then we call $m$ the real order of the curve. Obviously $m \geqslant n$, and the real order of an algebraic curve is not greater than its algebraic order.

A curve $K^{n+1}$ is a differentiable closed curve of real order $n + 1$ in $R_n$. It is a generalization of the algebraic curve of order $n + 1$ with one real branch.

All or all but one of the numbers of the characteristic $(a_0, a_1, \ldots, a_{n-1})$